

CONSTRUCTION OF THE MODULI SPACE OF Spin(7)-INSTANTONS

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ABSTRACT. We construct the moduli space of Spin(7)-instantons on a hermitian complex vector bundle over a closed 8-dimensional manifold endowed with a (possibly non-integrable) Spin(7)-structure. We find suitable perturbations that achieve regularity of the moduli space, so that it is smooth and of the expected dimension over the irreducible locus.

1. INTRODUCTION

Gauge theory in dimensions two, three and four is by now a classical research area in geometry and topology, which has been extensively studied and developed in the literature since the seventies, with formidable results in differential topology and other fields, see for example the book [8] and references therein.

Higher-dimensional gauge theory on the other hand is a much recent proposal appearing in the influential work of Donaldson and Thomas [9], and suggests studying a higher-dimensional version of the four-dimensional instanton equations which exists in the presence of the appropriate geometric structure. Higher-dimensional instantons had in fact already appeared *in disguise* in the physics literature as early as in the eighties [7, 45], although they were not systematically studied in the mathematical literature at the time. Recently, higher-dimensional gauge theory has experienced an increase in activity fueled both from pure mathematics, motivated by the program proposed by [9, 10, 42], as well as from string theory and in particular the Strominger system, see for example [6, 19, 20, 24, 36]. Early works in the topic include [15, 32, 38].

Most of the literature in higher-dimensional gauge theory, and in particular the one considering instantons in eight dimensions, focuses on manifolds of special holonomy. For example, Lewis [32] constructs Spin(7)-instantons on a resolution of T^8/F , where F is a finite subgroup acting on the 8-torus of T^8 , which has a Spin(7)-holonomy structure by the results of Joyce [25]. On the other hand, Tanaka [41] constructs examples of Spin(7)-instantons on the resolution of an appropriate Calabi-Yau four-orbifold quotiented by \mathbb{Z}_2 , which is a Spin(7)-holonomy manifold again by results of Joyce [26]. Walpuski [44] proves an existence theorem for a particular type of Spin(7)-instantons on Spin(7)-holonomy manifolds admitting appropriate local $K3$ Cayley fibrations.

Here instead, for reasons that will become apparent in a moment, we focus on 8-dimensional manifolds equipped with a generically non-integrable Spin(7)-structure. More concretely, in this article we initiate the construction of the moduli space of Spin(7)-instantons on a hermitian complex vector bundle E over a closed 8-manifold M equipped with a not necessarily integrable Spin(7)-structure, focusing on studying the transversality properties of the moduli space. The existence of a Spin(7)-structure on an 8-manifold is equivalent to the existence of a 4-form Ω point-wise satisfying a particular algebraic condition, but not involving any differential equation. In turn, Ω determines a Riemannian metric in a highly non-linear way, which is of Spin(7)-holonomy if and only if Ω is closed, in which case it defines a calibration [21]. In terms of Ω , the Spin(7)-instanton equation for a connection A in E is given by

$$*F_A = -F_A \wedge \Omega,$$

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where F_A is the curvature associated to A . We will refer to solutions of this equation as $\text{Spin}(7)$ -instantons. We are interested in studying the space of instantons modulo gauge transformations (automorphisms of the bundle). The $\text{Spin}(7)$ instanton condition modulo gauge transformations is elliptic regardless the integrability properties of the underlying $\text{Spin}(7)$ -structure. This is in clear contrast with the situation one encounters in 7-dimensions, and makes very natural working with arbitrary $\text{Spin}(7)$ -structures. In fact, the situation in 8-dimensions regarding $\text{Spin}(7)$ -instantons is similar in various aspects to that in 4-dimensions: for instance the deformation complex contains three terms [38], as it happens in 4-dimensions, being 8-dimensions the only case where this coincidence happens.

Aside from what we have already said, the motivations to consider $\text{Spin}(7)$ -instantons over closed 8-manifolds equipped with a not necessarily integrable $\text{Spin}(7)$ -structure are many. A practical reason comes from the existence of examples: explicit instances of closed 8-manifolds of $\text{Spin}(7)$ -holonomy are scarce [27]. Relaxing the integrability condition we are able to enlarge the available examples. In fact, we show that every Quaternionic-Kähler closed 8-manifold carries a necessarily non-integrable (and possibly unrelated) $\text{Spin}(7)$ -structure. This provides explicit examples of $\text{Spin}(7)$ -manifolds, such as $G_2/SO(4)$ and $\mathbb{H}P^2$.

As explained in [22], $\text{Spin}(7)$ -instantons can be useful to learn about the topology of 4-manifolds, if we are willing to take for granted a construction assigning to every 4-manifold X a $\text{Spin}(7)$ -manifold M_X . For example, M_X can be taken to be the total spinor bundle over X , which admits a $\text{Spin}(7)$ -structure which is generically non-integrable (it is integrable for example when $X = S^4$ [3]). By counting then $\text{Spin}(7)$ -instantons on M_X , one could in principle obtain an invariant for X .

Let us consider now the case that M is equipped with a Calabi-Yau structure (ω, θ) and an $SU(r)$ vector bundle E . Assuming $c_2(E) \in H^{2,2}(M)$, a $\text{Spin}(7)$ -instanton on E is equivalent to a polystable structure on E , a fact that was already noticed in [32] and that is used by Tian [43] to propose a way to attack the Hodge Conjecture by proving existence of $\text{Spin}(7)$ -instantons. This idea was explicitly considered and developed in reference [37], where Ramadas attempted to construct, without apparent success, $\text{Spin}(7)$ -instantons on some abelian varieties of Weil type for which Hodge Conjecture is yet to be settled. On the other hand, the first author [34, 35] studied, motivated by the previous proposal, under which conditions the existence of a polystable holomorphic structure on E for (M, ω, θ) implies the existence of a polystable holomorphic structure on E for $(M, \hat{\omega}, \hat{\theta})$, where $(\hat{\omega}, \hat{\theta})$ is an appropriately defined $\text{Spin}(7)$ -rotation of (ω, θ) .

Last but not least, there is a strong motivation coming from string theory to consider $\text{Spin}(7)$ -manifolds equipped with a non-integrable $\text{Spin}(7)$ -structure. The Strominger system [40] is a system of PDE's on a Riemannian manifold that encodes the conditions for it to be an admissible supersymmetric compactification background of Heterotic supergravity. When formulated in 8-dimensions, it involves a conformally balanced $\text{Spin}(7)$ -structure coupled to a $\text{Spin}(7)$ -instanton and a function [14, 17], and thus requires considering $\text{Spin}(7)$ -instantons with respect to generically non-integrable $\text{Spin}(7)$ -structures.

It should be noted¹ that using generically non-integrable $\text{Spin}(7)$ -structures has some drawbacks regarding the development of Donaldson's theory in eight dimensions. More concretely, in a $\text{Spin}(7)$ -holonomy manifold there is a topological bound in the L^2 -norm of the curvature of any $\text{Spin}(7)$ -instanton, which is an extra reason to hope that the corresponding $\text{Spin}(7)$ -instanton moduli space might admit a good compactification. However, for generic $\text{Spin}(7)$ -manifolds this is not longer true, and it is certainly possible that the L^2 -norm of the curvature goes to infinity as we move in the moduli space of $\text{Spin}(7)$ -instantons, leading to a *stronger* non-compactness which may indicate that such moduli space cannot be compactified and thus used to count invariants. This can be remedied by considering instead, as proposed in references [10, 28], a closed taming $\text{Spin}(7)$ -form. In this situation one again encounters a topological bound for the L^2 -norm of the curvature and then one can expect a moduli space admitting a nice compactification.

¹We thank Professor Dominic Joyce for explaining to us this important point.

Main results. The main goal of this work is to study transversality for the moduli space of $\text{Spin}(7)$ -instantons. We develop the theory *from scratch*. Section 2 contains the necessary background on $\text{Spin}(7)$ -representations as well as some results on the space of $\text{Spin}(7)$ -structures on an 8-dimensional vector space. Section 3 contains the necessary background on $\text{Spin}(7)$ manifolds. We prove that every closed 8-manifold admitting an almost-Quaternionic structure admits a (possibly unrelated) $\text{Spin}(7)$ -structure. Furthermore, we give a formula for the Dirac operator associated to the Ivanov connection which seems to be new in the literature. Section 4 contains a detailed analysis of some of the topological and analytic properties of the space of connections modulo gauge transformations. Little changes here from the situation in 4-dimensions and in fact we follow classical references on this. However, we have chosen to include explicitly all the pertinent results and proofs, mainly because some of them are not explicitly proven in the literature but also in order to give a systematic and complete exposition which can serve as the foundations for a theory of deformations of $\text{Spin}(7)$ -instantons. This section culminates with theorem 4.20, which among other things proves that the space of connections modulo gauge transformations is a topological Hausdorff space and gives a local description. Section 5 studies the local structure of the moduli space of $\text{Spin}(7)$ -instantons. The main result of this section is theorem 6.4 which gives the local model for the moduli space in terms of the hypercohomology groups of the appropriate deformation complex. Section 6 addresses the transversality properties of the moduli space of $\text{Spin}(7)$ -instantons on a complex hermitian vector bundle of any rank, considering first the rank-two case. In particular, theorem 6.6 proves that in the rank-two case, for a dense family of projector perturbations the corresponding moduli spaces are smooth at irreducible connections and of the expected dimension. In the higher-rank case, theorem 6.7 proves that for a dense family of holonomy perturbations, the perturbed moduli spaces are again smooth at irreducible connections and of the expected dimension. Section 7 considers explicitly the moduli space of $\text{Spin}(7)$ -instantons on a principal $\text{U}(1)$ -bundle. The main result of this section is theorem 7.2, which characterizes the moduli space of $\text{U}(1)$ -instantons for a generic $\text{Spin}(7)$ -structure under a relatively mild assumption on its torsion.

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2. REPRESENTATION THEORY OF THE GROUP $\text{Spin}(7)$

On \mathbb{R}^8 , with coordinates (x_1, \dots, x_8) , we consider the 4-form:

$$(1) \quad \begin{aligned} \Omega_0 = & dx_{1234} - dx_{1278} - dx_{1638} - dx_{1674} + dx_{1526} + dx_{1537} + dx_{1548} \\ & + dx_{5678} - dx_{5634} - dx_{5274} - dx_{5238} + dx_{3748} + dx_{2648} + dx_{2637}, \end{aligned}$$

where dx_{abcd} , $a, b, c, d = 1, \dots, 8$, stands for $dx_a \wedge dx_b \wedge dx_c \wedge dx_d$. The subgroup of $\text{GL}(8, \mathbb{R})$ that fixes Ω_0 is isomorphic to $\text{Spin}(7)$, which is a simply-connected, compact, Lie group of dimension 21, abstractly isomorphic to the double cover of $\text{SO}(7)$. This group also preserves the standard orientation of \mathbb{R}^8 and the euclidean metric, hence $\text{Spin}(7) \subset \text{SO}(8)$. Also, it is easy to see that $\Omega_0 = *\Omega_0$, where $*$ is the Hodge dual.

Consider an oriented 8-dimensional vector space V . A $\text{Spin}(7)$ -form is a 4-form Ω that can be written as Ω_0 in suitable coordinates, i.e., there exists an orientation-preserving isomorphism $f : V \rightarrow \mathbb{R}^8$ such that $\Omega = f^*\Omega_0$. The space \mathcal{S} of $\text{Spin}(7)$ -forms is thus a 43-dimensional homogeneous subspace of Λ^4 :

$$\mathcal{S} \cong \text{GL}^+(8, \mathbb{R}) / \text{Spin}(7).$$

The space \mathcal{S}_ν of $\text{Spin}(7)$ -forms compatible with a given volume form ν is diffeomorphic to the homogeneous space $\text{SL}(8, \mathbb{R}) / \text{Spin}(7)$, whereas the space \mathcal{S}_g of $\text{Spin}(7)$ -forms compatible with a given Riemannian structure g is diffeomorphic to the homogeneous space $\text{SO}(8) / \text{Spin}(7)$, which is of dimension 7. If on the other hand we only fix the conformal structure $c = [g]$, the corresponding space of compatible $\text{Spin}(7)$ -forms is $\mathcal{S}_c \cong (\mathbb{R}_+ \cdot \text{SO}(8)) / \text{Spin}(7)$.

There is a different characterization of $\text{Spin}(7)$ as the stabilizer of an element in one of the three irreducible representations of $\text{Spin}(8)$ on an eight-dimensional vector space V . For this, fix an orientation and a metric for V . Consider the Clifford algebra $\text{Cl}(8)$ associated to it, and recall that the group $\text{Spin}(8)$ satisfies $\text{Spin}(8) \subset \text{Cl}^{\text{even}}(8) \subset \text{Cl}(8)$. The 16-dimensional irreducible representation S of $\text{Cl}(8)$ admits a unique bilinear $\langle -, - \rangle : S \times S \rightarrow \mathbb{R}$ for which Clifford multiplication is orthogonal. It splits into two 8-dimensional irreducible inequivalent representations $S = S^+ \oplus S^-$ of $\text{Cl}^{\text{even}}(8)$. This produces two inequivalent representations $\gamma^\pm : \text{Spin}(8) \rightarrow \text{SO}(S^\pm)$ of $\text{Spin}(8)$. In addition, there is a third eight-dimensional representation of $\text{Spin}(8)$ given by its adjoint action on $V \hookrightarrow \text{Cl}(8)$, which we denote by $\text{Ad} : \text{Spin}(8) \rightarrow \text{SO}(V)$. The triality automorphism is an outer automorphism of $\text{Spin}(8)$ that permutes these three representations. All of the three representations are representations of the (universal) double cover of $\text{SO}(8)$. Clifford multiplication constitutes a $\text{Spin}(8)$ -equivariant map $c : V \otimes S^+ \rightarrow S^-$.

The group $\text{Spin}(7)$ can be now defined as the stabilizer of a unit-norm element η in either of S^+ , S^- or V . This gives three conjugacy classes of subgroups $\text{Spin}(7)$ inside $\text{Spin}(8)$, which are cyclically permuted by the triality automorphism of $\text{Spin}(8)$. The one that agrees with the previous definition is given by fixing an element $\eta \in S^7 \subset S^+$.

In the above notation, the standard representation $V = \mathbb{R}^8$ of $\text{SO}(8)$ induces the adjoint representation of $\text{Spin}(7)$. The positive spin representation of $\text{Spin}(8)$ splits in $\text{Spin}(7)$ -representations as:

$$S^+ = \langle \eta \rangle \oplus H,$$

where $\langle \eta \rangle$ denotes the one-dimensional trivial representation of $\text{Spin}(7)$ and H denotes the seven-dimensional representation isomorphic to the standard representation of $\text{SO}(7)$, induced by the adjoint representation of the double cover $\text{Spin}(7) \rightarrow \text{SO}(7)$. The $\text{Spin}(7)$ -equivariant map:

$$\begin{aligned} c : V \otimes S^+ &\rightarrow S^-, \\ v \otimes \eta &\mapsto v \cdot \eta, \end{aligned}$$

gives an isomorphism $V \rightarrow S^-$, via Clifford multiplication by η .

The $\text{Spin}(7)$ -representations on the exterior powers of V are as follows. Denote by $\Lambda^i = \Lambda^i V$, $i = 1, \dots, 8$. We have the following decompositions of the representation Λ^i into irreducible $\text{Spin}(7)$ factors [34]:

$$\begin{aligned} \Lambda^1 &= \Lambda_8^1, \\ \Lambda^2 &= \Lambda_7^2 \oplus \Lambda_{21}^2, \\ \Lambda^3 &= \Lambda_8^3 \oplus \Lambda_{48}^3, \\ \Lambda^4 &= \Lambda_+^4 \oplus \Lambda_-^4, \\ \Lambda_+^4 &= \Lambda_1^4 \oplus \Lambda_7^4 \oplus \Lambda_{27}^4, \\ \Lambda_-^4 &= \Lambda_{35}^4. \end{aligned}$$

Here Λ_j^i denotes the irreducible subrepresentation of Λ^i of dimension j . As mentioned above, $\Lambda_8^1 = V$, and $\Lambda_7^2 \cong H$. The second isomorphism is given by Clifford multiplication:

$$\begin{aligned} \mathcal{I} : \Lambda_7^2 &\longrightarrow H, \\ (2) \quad \alpha &\mapsto \alpha \cdot \eta, \end{aligned}$$

The fact that \mathcal{I} is an isomorphism follows from $\text{Spin}(7)$ -equivariance together with the identity $\langle \alpha \cdot \eta, \eta \rangle = 0$, for all $\alpha \in \Lambda^2$.

The subspace Λ_{21}^2 is the space associated to the Lie algebra $\mathfrak{spin}(7) \simeq \mathfrak{so}(7) \subset \mathfrak{so}(8) = \Lambda^2$. Moreover, we have the eigenvalue decomposition:

$$\begin{aligned} \Lambda_7^2 &= \{ \alpha \in \Lambda^2 \mid *(\alpha \wedge \Omega) = 3\alpha \}, \\ \Lambda_{21}^2 &= \{ \alpha \in \Lambda^2 \mid *(\alpha \wedge \Omega) = -\alpha \}. \end{aligned}$$

For three-forms, we have:

$$\begin{aligned}\Lambda_8^3 &= \{*(\alpha \wedge \Omega) | \alpha \in \Lambda_8^1\}, \\ \Lambda_{48}^3 &= \{\beta \in \Lambda^3 | \beta \wedge \Omega = 0\}.\end{aligned}$$

Regarding four-forms, we have that Λ_{\pm}^4 are the eigenspaces of the Hodge star operator $*$ on Λ^4 , both of dimension 35. It can be seen that $\Lambda_1^4 = \langle \Omega \rangle$. For describing Λ_7^4 , consider $\Lambda_7^2 \subset \Lambda^2 \subset V \otimes V \cong V \otimes V^*$, and take the image of $\Lambda_7^2 \otimes \Lambda_1^4 \rightarrow V \otimes V^* \otimes \Lambda^4 V \rightarrow \Lambda^4 V$, by contracting $(v \otimes \Theta) \otimes \alpha \mapsto v \wedge i_{\Theta} \alpha$.

Proposition 2.1. *Under the wedge map, we have the following:*

$$\begin{aligned}\Lambda_7^2 \times \Lambda_7^2 &\longrightarrow \Lambda_{27}^4 \oplus \Lambda_1^4, \\ \Lambda_7^2 \times \Lambda_{21}^2 &\longrightarrow \Lambda_7^4 \oplus \Lambda_{35}^4, \\ \Lambda_{21}^2 \times \Lambda_{21}^2 &\longrightarrow \Lambda_1^4 \oplus \Lambda_{27}^4 \oplus \Lambda_{35}^4.\end{aligned}$$

Proof. The first and second item appear in Subsection 2.1 and Remark 3 of [34].

The last one is proved in an analogous fashion to [34]. We consider the Spin(7)-structure induced by an SU(4)-structure under the inclusion $SU(4) \subset \text{Spin}(7)$. The SU(4)-structure is given by a complex structure J on V , a Kähler form ω and a $(4,0)$ -form θ , by setting $\Omega = \frac{1}{2}\omega^2 + \text{Re } \theta$. The complex structure allows to define the spaces of (p,q) -forms $\Lambda^{p,q} \subset \Lambda_{\mathbb{C}}^{p+q}$. We denote $\Delta^{p,q} = \text{Re}(\Lambda^{p,q})$, and use the sub-index *prim* to denote the subspace of primitive forms. By [34], $\Lambda_{21}^2 = A_- \oplus \Delta_{\text{prim}}^{1,1}$, where A_{\pm} are defined as the eigenspaces of $\Delta^{2,0}$ of the anti-linear complex Hodge operator $*_{\theta}$ defined by θ . To characterize the image of the wedge map, it is enough to see where it lies the image of $v \wedge v$ for an element v , since the collection of them span the image of $\Lambda_{21}^2 \times \Lambda_{21}^2$. For this we can use an element $v \in \Delta_{\text{prim}}^{1,1}$, which has image in $\Delta^{2,2} = \Delta_{\text{prim}}^{2,2} \oplus \Delta_{\text{prim}}^{1,1} \omega \oplus \langle \omega^2 \rangle$. Also, in [34] it is proved that $\Lambda_7^4 = A_- \omega \oplus \langle \text{im } \theta \rangle$, therefore all the three summands of $\Delta^{2,2}$ lie in $\Lambda_1^4 \oplus \Lambda_{27}^4 \oplus \Lambda_{35}^4$. To see that the three of them appear, note that the component of $v \wedge v$ in Λ_1^4 is non-zero, since $\text{Sym}^2(\Lambda_{21}^2)$ has an invariant quadratic form. Also it is easy to write a pair of forms in $\Delta_{\text{prim}}^{1,1}$ whose wedge is in $\Delta_{\text{prim}}^{2,2}$ (i.e. $dz_{1\bar{2}}, dz_{3\bar{4}}$), and also a pair of forms whose wedge is in $\Delta_{\text{prim}}^{1,1} \omega$ (i.e. $dz_{1\bar{1}} - \frac{1}{4}\omega$, and itself). So all three components appear. \square

Remark 2.2. If $\alpha \in \Lambda_7^2$, $\beta \in \Lambda_{21}^2$, and $\alpha \wedge \beta = 0$, then either $\alpha = 0$ or $\beta = 0$. To prove this, suppose that $\alpha \neq 0$. Then we can consider an SU(4) \subset Spin(7)-structure such that $\omega = \alpha$. With respect to this SU(4)-structure, we have $\Lambda_{21}^2 = A_- \oplus \Delta_{\text{prim}}^{1,1}$. But $\omega A_- \oplus \omega \Delta_{\text{prim}}^{1,1} \subset \Lambda^4$ is a 21-dimensional vector subspace. So $\omega \wedge \beta = 0 \implies \beta = 0$.

Now we want to have a closed look at the space $\mathcal{S} \subset \Lambda^4$ of all Spin(7)-forms, and the spaces $\mathcal{S}_g, \mathcal{S}_{\nu}, \mathcal{S}_c$ defined above.

Proposition 2.3. *Consider $\Omega \in \mathcal{S}$. We have the following tangent spaces at Ω :*

$$\begin{aligned}T_{\Omega} \mathcal{S} &= \Lambda_1^4 \oplus \Lambda_7^4 \oplus \Lambda_{35}^4, \\ T_{\Omega} \mathcal{S}_g &= \Lambda_7^4, \\ T_{\Omega} \mathcal{S}_{\nu} &= \Lambda_7^4 \oplus \Lambda_{35}^4, \\ T_{\Omega} \mathcal{S}_c &= \Lambda_1^4 \oplus \Lambda_7^4.\end{aligned}$$

Proof. The space \mathcal{S} is diffeomorphic to a homogeneous space $\mathcal{S} \cong \text{GL}^+(8, \mathbb{R}) / \text{Spin}(7)$, with Ω going to the neutral element. The tangent space $T_{\Omega} \mathcal{S}$ carries the isotropy representation of the stabilizer at Ω of the $\text{GL}^+(8, \mathbb{R})$ -action on \mathcal{S} , which is isomorphic to Spin(7). Hence the tangent space $T_{\Omega} \mathcal{S}$ becomes a Spin(7)-representation, and the action of Spin(7) on the four-forms correspond with the adjoint action on the tangent space $T_{\text{Id}}(\text{GL}^+(8, \mathbb{R}) / \text{Spin}(7))$. This means that $T_{\Omega} \mathcal{S}$ is a Spin(7)-subrepresentation of Λ^4 . As it is of dimension 43, the result follows. The other items are analogous. \square

3. Spin(7)-MANIFOLDS

Let M be an oriented 8-dimensional manifold. For each point $p \in M$ we denote by $\Sigma_p \subset \Lambda^4 T_p^* M$ the set of all Spin(7)-forms at p , namely $\Omega_p \in \Sigma_p$ if there exists an oriented isomorphism $f_p: T_p M \rightarrow \mathbb{R}^8$ such that $\Omega_p = f_p^* \Omega_0$, where Ω_0 is the canonical 4-form defined in equation (1). We denote by $\Sigma(M)$ the fiber bundle over M with fiber given by Σ_p , for each $p \in M$. Then, global sections of $\Sigma(M)$ are by construction in one-to-one correspondence with reductions of the frame bundle of M from $GL^+(8, \mathbb{R})$ to Spin(7). We define $\mathcal{S}(M) := \Omega^0(\Sigma(M))$ to be the space of smooth sections of Σ .

Definition 3.1. A Spin(7)-structure on an 8-dimensional manifold M is a reduction of the frame bundle $F(M)$ of M to Spin(7). That is, a Spin(7)-structure is a choice of a 4-form $\Omega \in \Omega^4(M)$ such that $\Omega_p \in \Sigma_p$, for each $p \in M$.

We have that $\mathcal{S}(M)$ is then the space of Spin(7)-structures on M . The existence of a Spin(7)-structure Ω on M allows for a point-wise decomposition of $\Lambda^i T^* M$, $i = 1, \dots, 8$, in Spin(7)-representations, which we denote with a subscript as described in section 2. We define

$$\Lambda_k^i(M) := \Lambda_k^i T^* M, \quad \Omega_k^i(M) := \Gamma(\Lambda_k^i T^* M),$$

where k denotes the specific Spin(7)-representation. A Spin(7)-structure determines an orientation and a riemannian metric on M . An oriented 8-dimensional manifold M admits a Spin(7)-structure (compatible with that orientation) if and only if M is spin and in addition (cf. Theorem 10.7 in [31])

$$(3) \quad p_1(M)^2 - 4p_2(M) + e(M) = 0,$$

where $p_1(M)$, $p_2(M)$ are the Pontrjagin classes of M and $e(M)$ is the Euler class of M .

Equivalently, we can characterize Spin(7)-structures by using non-zero spinors on M . We fix an orientation and a Riemannian metric, so that we have a frame bundle with structure group $SO(8)$. Recall that an eight-manifold is spin if and only if the frame bundle can be lifted to a Spin(8)-bundle $P_{\text{Spin}(8)}(M)$ in a compatible way with the double covering $\text{Ad}: \text{Spin}(8) \rightarrow SO(8)$. The obstruction for an orientable manifold to be spin is given by its second Stiefel-Whitney class. If this can be done for one Riemannian metric, then it can be done for any other metric. Assuming that M is spin, we can equip M with two spinor bundles S^\pm , which are associated to $P_{\text{Spin}(8)}(M)$ through the two eight-dimensional irreducible inequivalent representations $\gamma^\pm: \text{Spin}(8) \rightarrow SO(S^\pm)$ of Spin(8) and of chirality \pm .

Proposition 3.2. [31, Theorem 10.7] *An oriented 8-dimensional manifold M is Spin(7) if and only if it is spin and carries a unit-norm spinor $\eta \in \Gamma(S^+)$.*

We will call such unit spinor $\eta \in \Gamma(S^+)$ the *associated spinor* to the Spin(7)-structure Ω . as the The left hand side in equation (3) is equal to $e(S^+)$. Since the rank of S^+ equals the dimension of M , the existence of a nowhere zero spinor $\eta \in \Gamma(S^+)$ is equivalent to the vanishing $e(S^+) = 0$.

Let (M, Ω) be a Spin(7)-manifold, and let g be the induced Riemannian metric and ∇ the Levi-Civita connection. The Spin(7)-structure is integrable, that is, the holonomy of (M, g) is contained in $\text{Spin}(7) < SO(8)$, if and only if $\nabla \Omega = 0$. By [11, 13], this is equivalent to $d\Omega = 0$. Note that $*\Omega = \Omega$, so that in this case Ω is closed and co-closed. Examples of compact manifolds with Spin(7)-holonomy are relatively scarce. The first examples were given by D. Joyce [27]. If we relax the requirement of having Spin(7)-holonomy and we allow general Spin(7)-structures, then there are more examples. We find useful to point out the following result, which seems to have passed unnoticed in the literature.

Proposition 3.3. *Let M be a closed 8-manifold admitting an almost-Quaternionic structure on its tangent space. Then, M admits a Spin(7)-structure $\Omega \in \mathcal{S}(M)$, in general unrelated to the existent almost-Quaternionic structure.*

Proof. It follows from corollary 8.3 in [4] together with theorem 10.7 in [31]. \square

Proposition 3.3 automatically provides a relatively large number of explicit manifolds carrying Spin(7)-structures, which, to the best of our knowledge have not been studied or characterized.

In particular, proposition 3.3 shows that the eight-dimensional quaternionic space $\mathbb{H}\mathbb{P}^2$ carries a Spin(7)-structure. For an explicit early example of a compact eight-manifold carrying a non-integrable Spin(7)-structure the reader may consult [12].

For a Spin(7)-structure Ω , we define its torsion as [11]:

$$W := d\Omega \in \Omega^5(M).$$

As $\Omega^5(M) = \Omega_8^5(M) \oplus \Omega_{48}^5(M)$ in irreducible Spin(7)-representation, we have the orthogonal decomposition $W = W_8 \oplus W_{48}$, where $W_8 \in \Omega_8^5(M)$ and $W_{48} \in \Omega_{48}^5(M)$. We define the Lee form of the Spin(7)-structure as

$$\theta = *(d^*\Omega \wedge \Omega).$$

Then reference [11] distinguishes four types of Spin(7)-structures:

- Spin(7)-holonomy structures, defined by satisfying $W_8 = W_{48} = 0$.
- Balanced Spin(7)-structures, defined by having vanishing Lee-form, $\theta = 0$. So $W_8 = 0$.
- Locally conformally parallel Spin(7)-structures, defined by the condition $d\Omega = -\frac{1}{7}\theta \wedge \Omega$. So $W_{48} = 0$.
- Generic Spin(7)-structures, with no specific restriction on W_8 or W_{48} .

If we define a Spin(7)-structure via a unit spinor η as in proposition 3.2, then the Spin(7)-structure is integrable if and only if $\nabla\eta = 0$, where ∇ is the spin Levi-Civita connection. In the non-integrable case, there exists a canonical metric-compatible connection with torsion $\nabla^T: \Omega^0(S^+) \rightarrow \Omega^1(S^+)$ which preserves η , i.e., satisfies $\nabla^T\eta = 0$.

Theorem 3.4. *Let (M, Ω) be a Spin(7)-manifold with Spin(7)-structure Ω and associated spinor $\eta \in \Omega^0(S^+)$. The Cayley four-form Ω and the spinor η are related as follows:*

$$(4) \quad \eta \otimes \eta = 1 + \Omega + \nu,$$

where $*1 = \nu \in \Omega^8(M)$. In particular

$$(5) \quad \Omega(u, v, w, z) = \frac{1}{4!} \langle (u \wedge v \wedge w \wedge z) \cdot \eta, \eta \rangle, \quad u, v, w, z \in \mathfrak{X}(M).$$

Furthermore, there exists a unique connection ∇^T with fully antisymmetric torsion T such that $\nabla^T\eta = 0$. The torsion is given by

$$(6) \quad T = -d^*\Omega - *(\theta \wedge \Omega), \quad \theta = \frac{1}{6} * (d^*\Omega \wedge \Omega),$$

and it acts on η through Clifford multiplication as $T \cdot \eta = -\theta \cdot \eta$.

Proof. Equations (4) and (5) follow from [31, Theorem 10.18]. Equation (4) should be interpreted as follows: identifying $S \cong S^*$ by means of the bilinear product, we have an element $\eta \otimes \eta \in S \otimes S \cong S \otimes S^* = \text{End}(S) \cong \text{Cl}(M, g)$. But $\text{Cl}(M, g) \cong \Lambda^* T^*M$ as vector spaces. Then $\eta \otimes \eta$ is mapped to the poly-form $1 + \Omega + \nu$.

The fact that there exists a unique connection ∇^T satisfying $\nabla^T\eta = 0$ together with equation (6) and the Clifford action of T on η follow from [23, Theorem 1.1]. \square

Remark 3.5. The relation between T and W can be extracted from equation (6) and it is relatively involved:

$$*T = W + \frac{1}{6} (* (W \wedge \Omega) \wedge \Omega).$$

In particular, $T_8 = 0 \Leftrightarrow W_8 = 0$ and $T_{48} = 0 \Leftrightarrow W_{48} = 0$.

We will call ∇^T the *Ivanov connection* associated to the Spin(7)-structure Ω . For later use, we want to consider in more detail the properties of the Dirac operator associated to the Ivanov connection on the spinor bundle of a Spin(7)-manifold (M, Ω) as well as the corresponding index theorem. As usual, we will denote by $\eta \in \Gamma(S^+)$ the spinor corresponding the Spin(7)-structure Ω .

Remark 3.6. Let us describe the integrability condition in terms of $\nabla\eta$ and $\nabla\Omega$ and compare both tensors. First $\nabla\eta \in \Lambda^1 \otimes S^+$. But as it is $\langle \nabla\eta, \eta \rangle = 0$, we have $\nabla\eta \in \Lambda^1 \otimes H$. This produces an element $\mathcal{I}^{-1}(\nabla\eta) \in \Lambda^1 \otimes \Lambda_7^2$, via (2). To find it explicitly, note that the Ivanov connection is given by $\nabla_X^T Y = \nabla_X Y + \frac{1}{2}T(X, Y)$, in terms of the Levi-Civita connection. From this, it follows that $\nabla_X \eta = -\frac{1}{2}T(X, \cdot) \cdot \eta$. So $\mathcal{I}^{-1}(\nabla\eta) = -\frac{1}{2}T$. Note also that the wedge map (i.e., anti-symmetrization) gives an isomorphism $\Lambda^1 \otimes \Lambda_7^2 \rightarrow \Lambda^3$.

Second $\nabla\Omega \in \Lambda^1 \otimes \Lambda^4$. But $\nabla_X \Omega$ gives the variation of Ω_x , for x moving in the direction of X . As $T_\Omega \mathcal{S}_g = \Lambda_7^4$, by proposition 2.3, we have that $\nabla_X \Omega \in \Lambda_7^4$. So $\nabla\Omega \in \Lambda^1 \otimes \Lambda_7^4$. Again, the wedge map $\Lambda^1 \otimes \Lambda_7^4 \rightarrow \Lambda^5$ is an isomorphism, and $\nabla\Omega$ is mapped to $W = d\Omega$. In particular, we recover that $\nabla\Omega = 0 \Leftrightarrow d\Omega = 0$.

As we have already explained, since M is an 8-dimensional oriented spin manifold, the bundle of irreducible Clifford modules S admits the \mathbb{Z}_2 -grading $S = S^+ \oplus S^-$, given by the volume form ν , which is parallel, squares to plus one and it is central in $\text{Cl}^{\text{even}}(M, g)$. Let E be a real vector bundle over M . Then $S \otimes E = (S^+ \otimes E) \oplus (S^- \otimes E)$ automatically becomes a \mathbb{Z}_2 -graded bundle of real Clifford modules over M .

Let ∇_A be a connection on E and let ∇^T be the Ivanov spin connection. Associated to $\nabla = \nabla^T \otimes 1 + 1 \otimes \nabla_A$ we consider the Dirac operator

$$(7) \quad D_T^\pm : \Omega^0(S^\pm \otimes E) \rightarrow \Omega^0(S^\mp \otimes E).$$

By the Index theorem we have:

$$\text{Ind } D_T^+ = \text{Ind } D_T^- = \left\{ \text{ch } E \cdot \hat{A}(M) \right\} [M],$$

where $\text{ch } E$ denotes the Chern-character of E and $\hat{A}(M)$ denotes the A -roof genus of TM .

Let us recall that in a $\text{Spin}(7)$ -manifold with $\text{Spin}(7)$ -structure given by a positive-chirality spinor η the following isomorphisms hold

$$(8) \quad S^+ \cong \Omega^0(M) \oplus \Lambda_7^2(M), \quad S^- \cong \Lambda_8^1(M),$$

where $\Omega^0(M)$ is the trivial line bundle over M .

Proposition 3.7. *Through the isomorphisms (8), the Dirac operator D_T^- acts on $\Omega^1(M)$ as follows:*

$$\begin{aligned} D_T^- : \Omega^1(E) &\rightarrow \Omega^0(E) \oplus \Omega_7^2(E), \\ \tau &\mapsto d_A^* \tau \oplus \pi_7(d_A \tau + \iota_\tau T) \end{aligned}$$

where $\pi_7 : \Lambda^2 \rightarrow \Lambda_7^2$ is the orthogonal projection.

Proof. The isomorphism between $S^- \otimes E$ and $\Lambda^1 \otimes E$ is given by

$$\begin{aligned} F : \Omega^1(E) &\rightarrow \Omega^0(S^- \otimes E), \\ \tau &\mapsto \tau \cdot \eta. \end{aligned}$$

Therefore, for every $\Theta \in \Omega^0(S^- \otimes E)$ there exist a unique $\tau \in \Omega^1(E)$ such that $\Theta = \tau \cdot \eta$. Let $\{e^i\}$ be a local coframe and let $\{e_i\}$ be the corresponding local frame of TM . We have

$$\begin{aligned} (9) \quad D_T^-(\Theta) &= D_T^-(\tau \cdot \eta) \\ &= e^i \cdot \nabla_{e_i} \tau \cdot \eta \\ &= e^i \cdot \nabla_{e_i} \tau \cdot \eta \\ &= (e^i \wedge \nabla_{e_i}^T \tau) \cdot \eta - \langle e^i, \nabla_{e_i}^T \tau \rangle \eta \\ &= d_A^* \tau \cdot \eta + d_A \tau \cdot \eta + \iota_\tau T \cdot \eta. \end{aligned}$$

Here we have used that $\nabla^T \eta = 0$ as well as $d_A \tau = e^i \wedge \nabla_{e_i} \tau + \iota_\tau T$, and $d_A^* \tau = -\iota_{e_i} \nabla_{e_i} \tau$. The latter one needs that $i_{e_i} i_{e_i} i_\tau T = 0$, because T is fully antisymmetric. From equation (9) we finally obtain

$$D_T^-(\tau) = d_A^* \tau \oplus \pi_7(d_A \tau + \iota_\tau T),$$

for every $\tau \in \Omega^1(E)$. □

4. ANALYTIC PROPERTIES OF THE GROUP OF GAUGE TRANSFORMATIONS

Let (M, Ω) be an 8-dimensional manifold with an Spin(7)-structure Ω and let P be a principal G -bundle over M , where G is a compact, semi-simple Lie group whose Lie algebra we denote by \mathfrak{g} . Associated to P we consider a complex vector bundle $E = P \times_{\rho} \mathbb{C}^r$ of rank r , where ρ denotes a r -dimensional faithful irreducible complex representation of G . We denote by $\mathfrak{g}_E \subset \text{End}(E)$ the bundle of endomorphisms of E associated to the adjoint bundle of algebras $\text{ad}(P) = P \times_{\text{Ad}} \mathfrak{g}$ of P .

Remark 4.1. We will be mainly interested in the case $G = \text{U}(r)$, whose Lie algebra we denote by $\mathfrak{u}(r)$. In this case we will denote by $\mathfrak{u}_E \subset \text{End}(E)$ the bundle of skew-hermitian endomorphisms of E , whereas when necessary we will denote by $\mathfrak{su}_E \subset \text{End}(E)$ the bundle of trace-less skew-hermitian endomorphisms of E .

We will denote by \mathcal{A} the space of G -compatible connections on E . For $A \in \mathcal{A}$, we denote by $F_A \in \Omega^2(\mathfrak{g}_E)$ its curvature. In addition, we denote by $\pi_7 : \Lambda^2(M) \rightarrow \Lambda_7^2(M)$ and $\pi_{21} : \Lambda^2(M) \rightarrow \Lambda_{21}^2(M)$ the orthogonal projections onto the respective summands.

The group of gauge transformations \mathcal{G} is defined as the group of all differentiable automorphisms of E or, equivalently, as the space $\Omega^0(\text{Ad}(P))$ of all differentiable sections of the bundle $\text{Ad}(P) = P \times_{\text{Conj}} G$, where G acts on itself by conjugation. A third, equivalent, description of \mathcal{G} is given by

$$\mathcal{G} = \text{Map}_G(P, G),$$

i.e., by the space of differentiable maps from P to G which are G -equivariant with respect to the adjoint action of G on itself.

Definition 4.2. We define the **reduced gauge group** $\bar{\mathcal{G}}$ as $\bar{\mathcal{G}} = \mathcal{G}/Z(G)$, where $Z(G)$ denotes the center of G .

As it has been defined, the gauge group and the reduced gauge group have only the abstract structure of a group, not even a topological group. The gauge group can be made into a topological group by endowing it with the C^∞ compact-open topology. However, this is not the topology that we will use in this paper. In order to proceed further, we need to complete \mathcal{G} and \mathcal{A} using suitable Sobolev norms. These completions will induce the appropriate topological and metric structures on the corresponding completed spaces.

- We denote by $\Omega_{s+1}^0(\text{End } E)$ the Sobolev completion of $\Omega^0(\text{End } E)$ with respect to the Sobolev norm H_{s+1} . For $s > \frac{1}{2} \dim(M)$ the Sobolev continuous embedding theorem implies that $\Omega_{s+1}^0(\text{End } E) \subset C^0(\text{End } E)$ is a compact continuous embedding. Point-wise multiplication is well-defined and continuous in $\Omega_{s+1}^0(\text{End } E)$. We define \mathcal{G}_{s+1} to be the Sobolev completion of \mathcal{G} respect to the Sobolev norm H_{s+1} , obtained by considering \mathcal{G} as a subspace of $\Omega^0(\text{End } E)$. Hence $\mathcal{G}_{s+1} \subset \Omega_{s+1}^0(\text{End } E)$ as a closed subspace. We give \mathcal{G}_{s+1} the subspace topology induced by $\Omega_{s+1}^0(\text{End } E)$. For $s > \frac{1}{2} \dim M$ we have that \mathcal{G}_{s+1} is an infinite-dimensional smooth Hilbert-Lie group with respect to the topology given by the Sobolev norm H_{s+1} . In a similar way we Sobolev-complete $\bar{\mathcal{G}}$, obtaining $\bar{\mathcal{G}}_{s+1}$.
- As an infinite-dimensional Hilbert-Lie group, the Lie algebra $T_{\text{Id}}(\mathcal{G}_{s+1})$ of \mathcal{G}_{s+1} can be identified with the Sobolev completion $\Omega_{s+1}^0(\mathfrak{g}_E)$ of $\Omega^0(\mathfrak{g}_E)$ with respect to the Sobolev norm H_{s+1} . Hence $T_{\text{Id}}(\mathcal{G}_{s+1}) \cong \Omega_{s+1}^0(\mathfrak{g}_E)$.
- We define \mathcal{A}_s to be the Sobolev completion of \mathcal{A} with respect to the Sobolev-norm H_s . Fixing a base (smooth) connection $A_0 \in \mathcal{A}_s$ we can write:

$$\mathcal{A}_s = A_0 + \Omega_s^1(\mathfrak{g}_E).$$

Using Sobolev completions and taking $s > \frac{1}{2} \dim M$, which will assume henceforth, the action of \mathcal{G}_{s+1} on \mathcal{A}_s is smooth.

Remark 4.3. The previous remarks show that there are compact, continuous, embeddings of \mathcal{G}_{s+1} and \mathcal{A}_s respectively into the space of continuous sections $C^0(\text{End } E)$ of $\text{End } E$ and the space of continuous sections $C^0(T^*M)$ of T^*M . However, for applications to instantons we may need the

previous compact, continuous embeddings to be respectively in $C^2(\text{End } E)$ and $C^2(T^*M)$. This can be achieved simply by taking s to be large enough, for example $s > \dim(M)$.

The curvature operator

$$\mathcal{F}_s: \mathcal{A}_s \rightarrow \Omega_{s-1}^2(\mathfrak{g}_E), \quad \mathcal{F}_s(A) = F_A,$$

extends to a smooth, bounded, \mathcal{G}_{s+1} -equivariant map of infinite-dimensional Hilbert-spaces. There is a natural smooth action of \mathcal{G}_{s+1} on \mathcal{A}_s

$$\begin{aligned} \Phi_{s+1}: \mathcal{G}_{s+1} \times \mathcal{A}_s &\rightarrow \mathcal{A}_s, \\ (u, A) &\mapsto u \cdot A. \end{aligned}$$

The centre $Z(G)$ acts trivially on \mathcal{A}_s , so $\bar{\mathcal{G}}_s = \mathcal{G}_s/Z(G)$ also acts smoothly on \mathcal{A}_s . Let us fix a point $A \in \mathcal{A}_s$ and define $\Phi_{s+1}^A := \Phi_{s+1}(-, A): \mathcal{G}_{s+1} \rightarrow \mathcal{A}_s$. The derivative $(d\Phi_{s+1}^A)|_{\text{Id}}: \Omega_s^0(\mathfrak{g}_E) \rightarrow \Omega_s^1(\mathfrak{g}_E)$ of Φ_{s+1}^A at the identity $\text{Id} \in \mathcal{G}_{s+1}$ is given by

$$\begin{aligned} (d\Phi_{s+1}^A)|_{\text{Id}}: \Omega_{s+1}^0(\mathfrak{g}_E) &\rightarrow \Omega_s^1(\mathfrak{g}_E), \\ \tau &\mapsto -d_A \tau. \end{aligned}$$

Definition 4.4. We define the following spaces of connections modulo gauge transformations, equipped with the quotient topology

$$\mathcal{B}_s = \mathcal{A}_s/\mathcal{G}_{s+1} = \mathcal{A}_s/\bar{\mathcal{G}}_{s+1}.$$

Remark 4.5. We denote by $\mathcal{O}_A := \mathcal{G} \cdot A$ the orbit of the \mathcal{G}_{s+1} -action on \mathcal{A}_s passing through $A \in \mathcal{A}_s$. The tangent space of \mathcal{O}_A at A is given by

$$T_A \mathcal{O}_A = \{d_A \gamma \mid \gamma \in \Omega_{s+1}^0(\mathfrak{g}_E)\} \subset \Omega_s^1(\mathfrak{g}_E).$$

Let $A \in \mathcal{A}_s$. The stabilizer of A is defined as:

$$\Gamma_A = \{u \in \mathcal{G}_{s+1} \mid u \cdot A = A\}.$$

Elements in Γ_A correspond to covariantly constant automorphisms of E . The Lie algebra of Γ_A is given by

$$\mathfrak{t}_A = \text{Lie } \Gamma_A = \ker(d_A: \Omega_{s+1}^0(\mathfrak{g}_E) \rightarrow \Omega_s^1(\mathfrak{g}_E)).$$

We recall the following well-known lemma.

Lemma 4.6. *For any connection $A \in \mathcal{A}_s$, Γ_A is isomorphic to the centralizer of the holonomy H_A in G . In particular, Γ_A always contains the center $Z(G)$ of G .*

Remark 4.7. For $G = \text{SU}(r)$ we have $Z(\text{SU}(r)) = \mathbb{Z}_r$, and for $G = \text{U}(r)$, we have that $Z(\text{U}(r)) = \text{U}(1)$, the subgroup of diagonal matrices.

Definition 4.8. We say that a connection A is irreducible if the holonomy of A is equal to the structure group of P , i.e., $H_A = G$. We say it is reducible otherwise.

If a connection A is reducible, then the holonomy $H_A \subset G$ is strictly contained in G . Therefore the holonomy Lie algebra $\mathfrak{h}_A \subset \mathfrak{g}$ is strictly contained, and $F_A \in \Omega^2((\mathfrak{h}_A)_E)$.

Proposition 4.9. *Let $A \in \mathcal{A}_s$. If A is irreducible, then $\Gamma_A = Z(G)$ and the kernel of $d_A: \Omega_{s+1}^0(\mathfrak{g}_E) \rightarrow \Omega_s^1(\mathfrak{g}_E)$ is trivial (only constant sections). For $G = \text{SU}(r), \text{U}(r)$ and M simply-connected, then the above assertions are equivalent, and A is reducible if and only if there is a splitting $E = E_1 \oplus E_2$ with $A = A_1 \oplus A_2$.*

Proof. If A is irreducible, then $H_A = G$, so Γ_A is the centralizer of G , hence $\Gamma_A = Z(G)$. The Lie algebra of Γ_A is \mathfrak{t}_A , the kernel of d_A , hence \mathfrak{t}_A consists only of the constant sections.

Now assume that M is simply-connected and $G = \text{SU}(r), \text{U}(r)$. If $\Gamma_A \neq Z(G)$, then lemma 4.6 implies that the centralizer of H_A in G properly contains the center $Z(G)$ of G . Therefore H_A is properly contained in G and hence A is reducible. Also if \mathfrak{t}_A is non-trivial, then $Z(G) \subsetneq \Gamma_A$ because Γ_A has strictly bigger dimension than $Z(G)$, and the same holds. Suppose that $H_A \neq G$. Then by

simply-connectivity of M , H_A is connected. Being a subgroup of $U(r)$, it must be conjugated to a subgroup of some $U(k) \times U(r-k)$, $0 < k < r$. This implies that the bundle and the connection split as indicated. \square

We denote $\mathcal{A}_s^* \subset \mathcal{A}_s$ the subspace of irreducible connections on E , which is dense and open in \mathcal{A}_s .

Definition 4.10. We define the space of irreducible connections modulo gauge transformations, equipped with the quotient topology

$$\mathcal{B}_s^* = \mathcal{A}_s^* / \mathcal{G}_{s+1} = \mathcal{A}_s^* / \bar{\mathcal{G}}_{s+1} \subset \mathcal{B}_s.$$

Remark 4.11. The reduced gauge group $\bar{\mathcal{G}}_{s+1}$ acts freely on \mathcal{A}_s^* .

We define the following canonical projections

$$\pi: \mathcal{A}_{s+1} \rightarrow \mathcal{B}_s, \quad \pi: \mathcal{A}_{s+1}^* \rightarrow \mathcal{B}_s^*,$$

We proceed now to analyze the local structure of \mathcal{B}_s following references [1, 2, 8, 16, 30].

Lemma 4.12. *For any $s \geq 0$, the map $d_A: \Omega_{s+1}^0(\mathfrak{g}_E) \rightarrow \Omega_s^1(\mathfrak{g}_E)$ has finite-dimensional kernel and closed range. The kernel of d_A consists of C^∞ -sections of \mathfrak{g}_E and its dimension satisfies*

$$\dim \ker d_A \leq \text{rk } \mathfrak{g}_E.$$

Furthermore, there exists a constant c_{s+1} such that

$$(10) \quad \|\psi\|_{s+1} \leq c_{s+1} \|d_A \psi\|_s,$$

for all $\psi \perp \ker d_A$.

Proof. Elements $\phi \in \ker d_A$ are sections of a vector bundle parallel with respect to the connection d_A . Therefore, they are completely specified by their value at one point and hence we obtain $\dim \ker d_A \leq \text{rk } \mathfrak{g}_E$. Moreover $(\nabla_A)^{k+1} \phi = 0$ for all $k \geq 0$ and hence ϕ is C^∞ by the Sobolev embedding theorem.

Using now the identity $\|d_A \phi\|_s^2 + \|\phi\|_0^2 = \|\phi\|_{s+1}^2$ for all $\phi \in \Omega_{s+1}^0(\mathfrak{g}_E)$, equation (10) is equivalent to:

$$(11) \quad \|\psi\|_0^2 \leq c \|d_A \psi\|_0^2,$$

for some constant $c > 0$ and for all $\psi \perp \ker d_A$. We can now prove equation (11) by taking

$$c^{-1} = \inf \left\{ \left(\frac{\|d_A \psi\|_0^2}{\|\psi\|_0^2} \right) \mid \psi \perp \ker d_A \right\}.$$

In order to see that the constant c^{-1} given above is well-defined we just have to consider the following equalities

$$H_0(\Delta_A \psi, \psi) = H_0(d_A^* d_A \psi, \psi) = H_0(d_A \psi, d_A \psi) = \|d_A \psi\|_0^2.$$

Hence, c^{-1} is just the first non-zero eigenvalue of Δ_A . Since Δ_A is elliptic and M is closed, the spectrum of Δ_A is discrete and $\lambda > 0$. We conclude that $d_A: \Omega_{s+1}^0(\mathfrak{g}_E) \rightarrow \Omega_s^1(\mathfrak{g}_E)$ is bounded from below in the orthogonal of its null space and hence it has closed range. Alternatively, the fact that $d_A: \Omega_{s+1}^0(\mathfrak{g}_E) \rightarrow \Omega_s^1(\mathfrak{g}_E)$ has closed range follows from the orthogonal decomposition given in equation (23). \square

Lemma 4.13. *Let us assume that two connections $A_1, A_2 \in \mathcal{A}_s$ are equivalent by a gauge transformation $u \in \mathcal{G}_0$. Then $u \in \mathcal{G}_{s+1}$.*

Proof. Let us write $A_a = A + \tau_a$. Equation $A_1 = u \cdot A_2$ implies

$$d_A u = \tau_2 u - u \tau_1,$$

where the derivatives are understood in the weak sense. Note that $u \in \mathcal{G}_0$ means that u is $H_0 = L^2$. Taking now the $H_0 = L^2$ norm in the above equation,

$$\|d_A u\|_0 \leq \|\tau_2 u\|_0 + \|u \tau_1\|_0 \leq c (\|\tau_2\|_s \|u\|_0 + \|u\|_0 \|\tau_1\|_s),$$

where $c > 0$ is a positive constant. Here we have used the Sobolev multiplication theorem, which in our particular case gives the appropriate estimate for pointwise multiplication and states that $H_s \otimes H_i \rightarrow H_i$ is a continuous bilinear map for $0 \leq i \leq s$ provided that $s > \frac{1}{2} \dim(M)$. By the Sobolev embedding theorem, $H_1 \subset H_0$ and hence $\|u\|_0 < \infty$. Together with the fact that $\|\tau_a\|_s < \infty$ by assumption, we conclude

$$\|d_A u\|_0 < \infty.$$

Hence $u \in H_1$. Iteratively repeating the previous argument, we arrive to the last step

$$\|d_A u\|_s \leq \|\tau_2 u\|_s + \|u \tau_1\|_s \leq c(\|\tau_2\|_s \|u\|_s + \|u\|_s \|\tau_1\|_s),$$

which implies $\|d_A u\|_s < \infty$ and hence $u \in \mathcal{G}_{s+1}$. \square

Lemma 4.14. *Let $\{A_1^n\}$ and $\{A_2^n\}$ be sequences of points in \mathcal{A}_s that respectively converge to connections $A_1, A_2 \in \mathcal{A}_s$. Let us assume that $\{u_n\}$ is a sequence of points in $\mathcal{G}_0 \cap C_w^{s+1}(\text{End}(E))$ such that*

$$A_1^n = u_n \cdot A_2^n, \quad \forall n \in \mathbb{N}.$$

Then $\{u_n\} \subset \mathcal{G}_{s+1}$. Furthermore, after perhaps passing to a subsequence, $\{u_n\}$ converges to an element $u \in \mathcal{G}_{s+1}$ satisfying $A_1 = u \cdot A_2$.

Proof. The fact that u_n is in \mathcal{G}_{s+1} for all $n \in \mathbb{N}$ follows directly from lemma 4.13. We will assume then that $\{u_n\} \subset \mathcal{G}_{s+1}$. Let us write $A_a^n = A + \tau_a^n$, $a = 1, 2$. The sequences $\{\tau_a^n\}$ converge to $\tau_a \in \Omega_s^1(\mathfrak{g}_E)$ in H_s . Equation $A_1^n = u_n \cdot A_2^n$ implies

$$d_A u_n = \tau_2^n u_n - u_n \tau_1^n.$$

The sequences $\{\tau_1^n\}$ and $\{\tau_2^n\}$ converge in the H_s -norm, and hence they are uniformly bounded in $\Omega_s^1(\mathfrak{g}_E)$. Now, let us consider each u_n as a G -equivariant function on P and taking values on $G \subset \text{Mat}(r, \mathbb{C})$, which is a compact subspace of the vector space of square $r \times r$ complex matrices $\text{Mat}(r, \mathbb{C})$. Therefore $\{u_n\}$ is uniformly bounded in the L^∞ -norm. The uniform bound of $\{u_n\}$ in the L^∞ -norm implies a uniform bound in the L^2 -norm by compactness of M . The Sobolev multiplication theorem implies now that $H_s \times H_i \rightarrow H_i$, $i = 0, \dots, s$, is a continuous bilinear map and in addition gives us the following estimate

$$\|d_A u_n\|_0 = \|\tau_2^n u_n - u_n \tau_1^n\|_0 \leq c(\|\tau_2^n\|_s \|u_n\|_0 + \|u_n\|_0 \|\tau_1^n\|_s),$$

for an appropriate constant $c > 0$. Hence, we conclude that $\{d_A u_n\}$ is uniformly bounded with respect to L^2 , implying that $\{u_n\}$ is uniformly bounded in H_1 . Iteratively repeating this process, we arrive to the last step

$$\|d_A u_n\|_s = \|\tau_2^n u_n - u_n \tau_1^n\|_s \leq c(\|\tau_2^n\|_s \|u_n\|_s + \|u_n\|_s \|\tau_1^n\|_s),$$

for a constant $c > 0$. Hence, we conclude that $\{d_A u_n\}$ is uniformly bounded with respect to H_s , implying that $\{u_n\}$ is uniformly bounded in H_{s+1} . Hence, perhaps after passing to a subsequence, $\{u_n\}$ weakly converges in the H_{s+1} -norm. Since $\{u_n\} \subset \mathcal{G}_{s+1}$ and \mathcal{G}_{s+1} is a closed subspace of $\Omega_s^0(\text{End}(E))$ the weak limit u of $\{u_n\}$ is in \mathcal{G}_{s+1} . We want to prove now that in fact $\{u_n\}$ strongly converges to u . By strict inequality of the Sobolev embedding theorems, we obtain that the embedding

$$L_{s+1}^2 \hookrightarrow L_s^2,$$

is a compact map. Hence, $\{u_n\}$ strongly converges in H_s . Using now that the Sobolev multiplication theorem implies that $H_s \times H_s \rightarrow H_s$ is continuous we conclude that $\{\tau_2^n u_n - u_n \tau_1^n\}$ and hence $\{d_A u_n\}$ strongly converges in H_s , whence $\{u_n\}$ strongly converges in H_{s+1} . Furthermore, by uniqueness of limits we obtain

$$d_A u = \tau_2 u - u \tau_1.$$

We have proven that, perhaps after passing to a subsequence, $\{u_n\}$ converges to u in H_{s+1} and in particular, $u \in \mathcal{G}_{s+1}$. Finally, from $d_A u = \tau_2 u - u \tau_1$ follows that $A_1 = u \cdot A_2$ and we conclude. \square

Let $A \in \mathcal{A}_s$. We define

$$(12) \quad T_{A,\epsilon} := \{A + \tau, \tau \in \Omega_s^1(\mathfrak{g}_E) \mid d_A^* \tau = 0, \|\tau\|_s < \epsilon\} \subset \mathcal{A}_s.$$

Lemma 4.15. *For each $A \in \mathcal{A}_s$, the stabilizer subgroup $\Gamma_A \subset \mathcal{G}_{s+1}$ fixes $T_{A,\epsilon}$.*

Proof. Follows from invariance of A under Γ_A together with the fact that, for all $u \in \Gamma_A$, we have $d_A^*(u \cdot \tau) = u \cdot d_A^* \tau$ and

$$\|u \cdot \tau\|_s = \|u \tau u^{-1}\|_s = \|\tau\|_s < \epsilon.$$

□

Lemma 4.16. *Let $A \in \mathcal{A}_s$. There exists a $\epsilon > 0$ such that if $A_1, A_2 \in T_{A,\epsilon}$ are connections equivalent by a gauge transformation $u \in \mathcal{G}_{s+1}$ with $\|u - \text{Id}\|_{s+1} < \epsilon$, then $A_1 = \gamma \cdot A_2$ for some $\gamma \in \Gamma_A \subset \mathcal{G}_{s+1}$.*

Proof. By lemma 4.12, the bounded linear map $d_A: \Omega_{s+1}^0(\mathfrak{g}_E) \rightarrow \Omega_s^1(\mathfrak{g}_E)$ has closed range. Therefore, there exists the following orthogonal decomposition:

$$(13) \quad T_A \mathcal{A}_s \cong \Omega_s^1(\mathfrak{g}_E) \cong \text{im}(d_A) \oplus \ker(d_A^*),$$

where d_A^* is the adjoint of d_A with respect to H_s . This adjoint is compatible with the standard L^2 -adjoint. Notice that $\text{im}(d_A)$ is the tangent space to the orbit \mathcal{O}_A at A . We define now the following differentiable map of smooth Hilbert manifolds

$$\begin{aligned} \Psi: \mathcal{G}_{s+1} \times T_{A,\epsilon} &\rightarrow \mathcal{A}_s, \\ (u, \tau) &\mapsto u \cdot \tau. \end{aligned}$$

The differential of Ψ at $(\text{Id}, 0) \in \mathcal{G}_{s+1} \times T_{A,\epsilon}$ is given by:

$$\begin{aligned} d\Psi_{(\text{Id},0)}: \Omega_{s+1}^0(\mathfrak{g}_E) \times \ker d_A^* &\rightarrow \Omega_s^1(\mathfrak{g}_E), \\ (l, \tau) &\mapsto -d_A l + \tau, \end{aligned}$$

where we have identified $T_{\text{Id}}(\mathcal{G}_{s+1}) \cong \Omega_{s+1}^0(\mathfrak{g}_E)$ and $T_A(T_{A,\epsilon}) \cong \ker(d_A^*)$. With respect to the splitting given in equation (13) we can write equation $d\Psi_{(\text{Id},0)}$ as follows

$$d\Psi_{(\text{Id},0)} = (-d_A, \text{Id}),$$

and hence we conclude that $d\Psi_{(\text{Id},0)}$ is an isomorphism of Banach spaces if and only if d_A is injective.

For clarity we distinguish now three cases, although strictly speaking each case is a particular case of the next one.

- $A \in \mathcal{A}_s^*$ **irreducible and** $Z(G) = \{\text{Id}\}$. In this case $d_A: \Omega_{s+1}^0(\mathfrak{g}_E) \rightarrow \Omega_s^1(\mathfrak{g}_E)$ is injective (it has trivial kernel), and hence $d\Psi_{(\text{Id},0)}$ is an isomorphism. It follows then by the inverse function theorem that there exists a neighborhood

$$\mathcal{N}_{\text{Id},\epsilon} := \{u \in \mathcal{G}_{s+1} \mid \|u - \text{Id}\|_{s+1} < \epsilon\},$$

of $\text{Id} \in \mathcal{G}_{s+1}$ and a neighborhood $U(A)$ of $A \in \mathcal{A}_s^*$ such that $\Psi: \mathcal{N}_{\text{Id},\epsilon} \times T_{A,\epsilon} \rightarrow U(A) \subset \mathcal{A}_s^*$ is a diffeomorphism. Therefore, any connection in $A' \in U(A)$ can be written as $A' = u \cdot A$ for the appropriate $u \in \mathcal{N}_{\text{Id},\epsilon}$ in a neighborhood of the identity. In particular, for the ϵ appearing in $\mathcal{N}_{\text{Id},\epsilon}$, if $A_1, A_2 \in T_{A,\epsilon}$ such that $A_1 = u \cdot A_2$ for a gauge transformation $u \in \mathcal{G}_{s+1}$ satisfying $\|u - \text{Id}\|_{s+1} < \epsilon$ then $u = \text{Id}$ and $A_1 = A_2$.

- $A \in \mathcal{A}_s^*$ **irreducible and any** $Z(G)$. In this case $d_A: \Omega_{s+1}^0(\mathfrak{g}_E) \rightarrow \Omega_s^1(\mathfrak{g}_E)$ may have kernel, isomorphic to $\zeta(\mathfrak{g})$, the Lie algebra of $Z(G)$. Therefore, we will slightly modify the domain of the map Ψ in order to obtain a diffeomorphism. Instead of $\mathcal{G}_{s+1} \times T_{A,\epsilon}$ we consider

$$(\mathcal{G}_{s+1} \times T_{A,\epsilon})/Z(G) \cong \bar{\mathcal{G}}_{s+1} \times T_{A,\epsilon}.$$

Here $Z(G)$ acts trivially on $T_{A,\epsilon}$. Since $Z(G) \subset \mathcal{G}_{s+1}$ is a normal Hilbert subgroup of \mathcal{G}_{s+1} , we have that $\mathcal{G}_{s+1}/C(G)$ is again an infinite-dimensional Hilbert Lie group with Lie algebra isomorphic to $\Omega_{s+1}^0(\mathfrak{g}_E)/\zeta(\mathfrak{g})$ (cf.[18]). This isomorphism follows simply from the fact that

$Z(G)$ is the subgroup of \mathcal{G}_{s+1} consisting on parallel sections of the endomorphism bundle with respect to d_A . We define then the following differentiable map

$$\begin{aligned}\bar{\Psi}: \bar{\mathcal{G}}_{s+1} \times T_{A,\epsilon} &\rightarrow \mathcal{A}_s, \\ (u, \tau) &\mapsto u \cdot \tau.\end{aligned}$$

Notice that $\bar{\Psi}$ is well-defined since different representatives of an element in $\bar{\mathcal{G}}_{s+1}$ differ by an element in $Z(G)$ which does not affect $\tau \in \ker d_A^*$. The differential $d\bar{\Psi}_{(\text{Id},0)}$ of $\bar{\Psi}$ at $(\text{Id}, 0)$ is given by

$$\begin{aligned}d\bar{\Psi}_{(\text{Id},0)}: \frac{\Omega_{s+1}^0(\mathfrak{g}_E)}{\zeta(\mathfrak{g})} \times \ker d_A^* &\rightarrow \Omega_s^1(\mathfrak{g}_E), \\ (l, \tau) &\mapsto -d_A l + \tau,\end{aligned}$$

Clearly now d_A is injective and therefore $d\bar{\Psi}_{(\text{Id},0)}$ is an isomorphism of infinite-dimensional Banach spaces. We conclude that there exists a neighborhood

$$\bar{\mathcal{N}}_{\text{Id},\epsilon} := \{u \in \bar{\mathcal{G}}_{s+1} \mid \|u - \text{Id}\|_{s+1} < \epsilon\},$$

of $\text{Id} \in \bar{\mathcal{G}}_{s+1}$ and a neighborhood $U(A)$ of $A \in \mathcal{A}_s^*$ such that $\bar{\Psi}: \bar{\mathcal{N}}_{\text{Id},\epsilon} \times T_{A,\epsilon} \rightarrow U(A) \subset \mathcal{A}_s^*$ is a diffeomorphism.

- **$A \in \mathcal{A}_s$ not necessarily irreducible.** In this case the stabilizer Γ_A of A may be non-trivial, and d_A can have non-trivial kernel, which is isomorphic to the Lie algebra \mathfrak{t}_A of Γ_A . We again slightly modify the definition of Ψ and instead define the following differentiable map

$$\begin{aligned}\bar{\Psi}: (\mathcal{G}_{s+1} \times T_{A,\epsilon})/\Gamma_A &\rightarrow \mathcal{A}_s, \\ [u, \tau] &\mapsto u \cdot \tau.\end{aligned}$$

Here $[u, \tau]$ denotes the equivalence class of (u, τ) respect to the action of Γ_A . We first check that $\bar{\Psi}$ is well-defined, namely that its value does not depend on the representative. If $(u, \tau) \in (\mathcal{G}_{s+1} \times T_{A,\epsilon})/\Gamma_A$ then any other representative is of the form $(u\gamma^{-1}, \gamma\tau\gamma^{-1})$ for a unique $\gamma \in \Gamma_A$. It is then a direct computation to check that $\bar{\Psi}(u, \tau) = \bar{\Psi}(u\gamma^{-1}, \gamma\tau\gamma^{-1})$.

The action of Γ_A on $\mathcal{G}_s \times T_{A,\epsilon}$ is free, since it is free in the first variable. Therefore $(\mathcal{G}_{s+1} \times T_{A,\epsilon})/\Gamma_A$ is smooth and the tangent space at $(\text{Id}, 0)$ is $\frac{\Omega_{s+1}^0(\mathfrak{g}_E)}{\mathfrak{t}_A} \times \ker d_A^*$. The differential $d\bar{\Psi}_{(\text{Id},0)}$ of $\bar{\Psi}$ at $(\text{Id}, 0)$ is

$$\begin{aligned}d\bar{\Psi}_{(\text{Id},0)}: \frac{\Omega_{s+1}^0(\mathfrak{g}_E)}{\mathfrak{t}_A} \times \ker d_A^* &\rightarrow \Omega_s^1(\mathfrak{g}_E), \\ (l, \tau) &\mapsto -d_A l + \tau.\end{aligned}$$

Clearly now d_A is injective and therefore $d\bar{\Psi}_{(\text{Id},0)}$ is an isomorphism of infinite-dimensional Banach spaces. We conclude that there exists a neighborhood

$$\bar{\mathcal{N}}_{\text{Id},\epsilon} := \{u \in \bar{\mathcal{G}}_{s+1} \mid \|u - \text{Id}\|_{s+1} < \epsilon\},$$

of $\text{Id} \in \bar{\mathcal{G}}_{s+1}$ and a neighborhood $U(A)$ of $A \in \mathcal{A}_s^*$ such that $\bar{\Psi}: (\bar{\mathcal{N}}_{\text{Id},\epsilon} \times T_{A,\epsilon})/\Gamma_A \rightarrow U(A) \subset \mathcal{A}_s^*$ is a diffeomorphism.

Therefore, any connection in $A' \in U(A)$ can be written as $A' = u \cdot A$ for the appropriate $u \in \bar{\mathcal{N}}_{\text{Id},\epsilon}$ in a neighborhood of the identity. In particular, if $A_1, A_2 \in T_{A,\epsilon}$ satisfy $A_1 = u \cdot A_2$ for a $u \in \bar{\mathcal{N}}_{\text{Id},\epsilon}$ then $A_1 = \gamma \cdot A_2$ for a $\gamma \in \Gamma_A$.

□

The previous lemma shows that $T_{A,\epsilon}$ is a local slice for the action of a local neighborhood of the identity in \mathcal{G}_{s+1} . We want to show now that, perhaps after taking a smaller $\epsilon > 0$, $T_{A,\epsilon}/\Gamma_A$ is a slice for the complete gauge group \mathcal{G}_{s+1} .

Lemma 4.17. *Let $A \in \mathcal{A}_s$. There exists a $\epsilon > 0$ such that if $A_1, A_2 \in T_{A,\epsilon}$ are equivalent by a gauge transformation $u \in \mathcal{G}_{s+1}$, then $u = \gamma \in \Gamma_A$ and hence $A_1 = \gamma \cdot A_2$.*

Proof. Lemma 4.16 shows that

$$\begin{aligned} \Psi: (\mathcal{G}_{s+1} \times T_{A,\epsilon}) / \Gamma_A &\rightarrow \mathcal{A}_s, \\ [u, \tau] &\mapsto u \cdot \tau, \end{aligned}$$

is a local diffeomorphism. We want to show that there exists an $\epsilon > 0$ such that Ψ is a global diffeomorphism from $(\mathcal{G}_{s+1} \times T_{A,\epsilon}) / \Gamma_A$ onto its image. If there was not such $\epsilon > 0$, there would exist sequences $\{A_1^n\}, \{A_2^n\} \in T_{A,\epsilon}$ and $\{u_n\} \in \mathcal{G}_{s+1}$ such that

$$(14) \quad A_1^n = u_n \cdot A_2^n, \quad \lim_{n \rightarrow \infty} A_1^n = \lim_{n \rightarrow \infty} A_2^n = A, \quad [A_1^n] \neq [A_2^n].$$

By lemma 4.14 we can extract a subsequence of $\{u_n\}$ such that it converges to $u \in \mathcal{G}_{s+1}$ and satisfies $A = u \cdot A$. Therefore $u \in \Gamma_A$. Hence, the sequences $[A_1^n, \text{Id}]$ and $[A_2^n, u_n]$ in $(T_{A,\epsilon} \times \mathcal{G}_{s+1}) / \Gamma_A$ both converge to $[A, \text{Id}]$. By the local diffeomorphism property of Ψ this implies that for $n > n_0$ for some fixed n_0 we have $[A_1^n, \text{Id}] = [A_2^n, u_n]$, contradicting the third equation in (14). \square

From the previous lemma we obtain the following corollary.

Corollary 4.18. *For every fixed $A \in \mathcal{A}_s$ there exists a polynomial $p(x, y)$ such that if $A_1 = A + \tau_1$ and $A_2 = A + \tau_2$ satisfy $u \cdot A_2 = A_1$ for some $u \in \mathcal{G}_{s+1}$ then*

$$\|u\|_{s+1} \leq p(\|\tau_1\|_s, \|\tau_2\|_s).$$

Lemmas 4.16 and 4.17 finally prove that $T_{A,\epsilon}$ is a slice for the gauge group \mathcal{G}_{s+1} acting on the space of connections \mathcal{A}_s .

Corollary 4.19. *For small enough $\epsilon > 0$ and $A \in \mathcal{A}_s$, $T_{A,\epsilon} / \Gamma_A$ is a local slice for the action of \mathcal{G}_{s+1} on \mathcal{A}_s .*

Theorem 4.20. *The following statements hold:*

- The space \mathcal{B}_s is a Hausdorff topological space.
- The subspace $\mathcal{B}_s^* \subset \mathcal{B}_s$ is an open in \mathcal{B}_s .
- The space \mathcal{B}_s^* is a smooth Hilbert manifold with local charts given by $\pi: T_{A,\epsilon} \rightarrow \mathcal{B}_s^*$ for $\epsilon > 0$ small enough.
- The map $\pi: \mathcal{A}_s^* \rightarrow \mathcal{B}_s^*$ is a smooth principal bundle.
- For each $A \in \mathcal{A}_s^*$, the stabilizer $\Gamma_A \subset \mathcal{G}_{s+1}$ preserves $T_{A,\epsilon}$ and the map

$$h: T_{A,\epsilon} / \Gamma_A \rightarrow \mathcal{B}_s,$$

is a homeomorphism onto a neighborhood of $h([A])$, which in addition is a diffeomorphism outside the fixed point set of Γ_A .

Proof. The fact that the projection $\pi: \mathcal{A}_s \rightarrow \mathcal{B}_s$ is open when \mathcal{B}_s is equipped with the quotient topology together with the fact that \mathcal{A}_s is first countable imply that \mathcal{B}_s is also first countable. Hence \mathcal{B}_s is Hausdorff if and only if every convergent sequence has a unique limit. Let us assume then that there exists a convergent sequence with two different limits. In other words, we assume that there exist sequences $\{A_1^n\}$ and $\{A_2^n\}$ of connections such that for all $n \in \mathbb{N}$ we have

$$A_1^n = u_n \cdot A_2^n, \quad u_n \in \mathcal{G}_{s+1},$$

and such that $A_1 = \lim_{n \rightarrow \infty} A_1^n$ and $A_2 = \lim_{n \rightarrow \infty} A_2^n$ are gauge-inequivalent connections. We set $A_2 = A_1 + \tau_2$, $A_2^n = A_1 + \tau_2^n$ and $A_1^n = A_1 + \tau_1^n$, where $\tau_2, \tau_2^n, \tau_1^n \in \Omega_s^1(\mathfrak{g}_E)$ for all $n \in \mathbb{N}$. By hypothesis we have $A_1 = \lim_{n \rightarrow \infty} A_1^n$ and hence $\lim_{n \rightarrow \infty} \tau_1^n = 0$ and $\lim_{n \rightarrow \infty} \tau_2^n = \tau_2$, in the Sobolev H_s -norm. In particular $\|\tau_1^n\|_s$ and $\|\tau_2^n\|_s$ are uniformly bounded, which implies, using corollary 4.18, that there exists a constant $c > 0$ such that $\|u_n\|_{s+1} < c$ for all $n \in \mathbb{N}$. Applying now the compact Sobolev embedding theorem we conclude that there is a subsequence of $\{u_n\}$ which converges strongly to an element $u \in \mathcal{G}_{s+1}$ in the Sobolev H_s -norm. Using now that

$$d_A u_n = u_n \tau_1^n - \tau_2^n u_n, \quad \forall n \in \mathbb{N},$$

we obtain

$$\|d_{A_1} u_n\|_s \leq c(\|u_n\|_s \|\tau_1^n\|_s + \|\tau_2^n\|_s \|u_n\|_s),$$

and hence we conclude that $\{u_n\}$ converges to $u \in \mathcal{G}_{s+1}$ in the H_{s+1} -norm. Therefore, $A_1 = u \cdot A_2$ and every convergent sequence in \mathcal{B}_s has a unique limit, which in turn proves that \mathcal{B}_s is Hausdorff.

In order to prove that $\mathcal{B}_s^* \subset \mathcal{B}_s$ is open we prove that for every $[A] \in \mathcal{B}_s^*$ there exists a neighborhood $U([A]) \subset \mathcal{B}_s^*$ of $[A]$ contained in \mathcal{B}_s^* . Let us assume otherwise. Then, every neighborhood of $[A]$ contains a reducible connection and, since \mathcal{B}_s is first countable, this implies the existence of a sequence $\{[A_n^R]\}$ of reducible connections in \mathcal{B}_s converging to $[A] \in \mathcal{B}_s^*$. The definition of reducibility means that the holonomy Lie algebra $\mathfrak{h}_{A_n^R}$ is not the total space. Passing to a subsequence, we may assume that $\mathfrak{h}_{A_n^R}$ tends to a subspace $\mathfrak{h} \subset \mathfrak{g}$. So $F_A = \lim_{n \rightarrow \infty} F_{A_n^R} \in \Omega^2(\mathfrak{h})$. We take $s > 0$ large enough so that the connections are C^1 , and hence F_A is C^0 . This implies that A has to be reducible.

Corollary 4.19 implies that every $A \in \mathcal{B}_s^*$ has a neighborhood homeomorphic to $T_{A,\epsilon}$, which is an infinite-dimensional Hilbert space. The fact that on overlapping open sets in \mathcal{B}_s^* the corresponding changes of coordinates are smooth, as well as the fact that $\pi : \mathcal{A}^* \rightarrow \mathcal{B}_s^*$ is a smooth principal bundle are both proven in great detail in [33]. The rest of the statements follow now from lemma 4.15 and corollary 4.19. \square

Remark 4.21. The principal fibration

$$\bar{\mathcal{G}}_{s+1} \rightarrow \mathcal{A}_s^* \rightarrow \mathcal{B}_s^*$$

induces a long exact sequence in homotopy which implies that

$$\pi_{k+1}(\mathcal{B}_s^*) = \pi_k(\bar{\mathcal{G}}_{s+1}), \quad k \geq 0.$$

In particular $\pi_1(\mathcal{B}_s^*) = \pi_0(\bar{\mathcal{G}}_{s+1})$.

5. LOCAL ANALYSIS OF THE MODULI SPACE OF $\text{Spin}(7)$ -INSTANTONS

As in the previous section, let (M, Ω) be an 8-dimensional manifold with an $\text{Spin}(7)$ -structure Ω and let P be a principal G -bundle over M , where G is a compact semi-simple Lie group. Associated to P we consider a complex vector bundle $E = P \times_{\rho} \mathbb{C}^r$ of rank r , where ρ denotes an r -dimensional faithful irreducible complex representation of G . As explained in Section 2, there is a decomposition $\Lambda^2 = \Lambda_7^2 \oplus \Lambda_{21}^2$ and projections $\pi_7 : \Lambda^2 \rightarrow \Lambda_7^2$ and $\pi_{21} : \Lambda^2 \rightarrow \Lambda_{21}^2$. The following is the central object of this paper.

Definition 5.1. [43] A connection $A \in \mathcal{A}_s$ is a $\text{Spin}(7)$ -instanton if $\pi_7(F_A) = 0$.

Remark 5.2. The $\text{Spin}(7)$ -instanton equation $\pi_7(F_A) = 0$ is equivalent to

$$(15) \quad *F_A = -\Omega \wedge F_A,$$

which is a first-order equation on the connection $A \in \mathcal{A}$. It implies a second-order equations on A . Acting with d_A on (15) we obtain, using the Bianchi identity $d_A F_A = 0$,

$$(16) \quad d_A^* F_A = -*(F_A \wedge W),$$

where $W = d\Omega$ is the torsion of the $\text{Spin}(7)$ -structure. Interestingly enough, equation (16) follows from the following action, which generalizes the standard Yang-Mills action of a connection

$$(17) \quad S(A) = \int_M (\kappa(F_A \wedge *F_A) + \kappa(F_A \wedge F_A) \wedge \Omega),$$

where κ is the bilinear form induced on the adjoint bundle \mathfrak{g}_E by the Killing form of \mathfrak{g} . For a $\text{Spin}(7)$ -holonomy manifold (M, Ω) , the second term in (17) is topological and we (classically) obtain the standard Yang-Mills action. For a general $\text{Spin}(7)$ -manifold (M, Ω) , not necessarily of $\text{Spin}(7)$ -holonomy, this term is not topological and does indeed contribute to the equations of motion. To the knowledge of the authors, the physical interpretation of (17) is still open. For example, it would be interesting to see if it can be supersymmetrized or obtained by dimensional reduction from a supersymmetric Yang-Mills theory on a curved ten-dimensional background.

Remark 5.3. The Spin(7)-instanton equation can be written in terms of the spinor η defining the Spin(7)-structure on M . The condition $\pi_7(F_A) = 0$ is equivalent to

$$(18) \quad F_A \cdot \eta = 0.$$

We are interested in studying the *moduli space* of Spin(7)-instantons on E , namely the space of connections in \mathcal{A} satisfying (15) modulo gauge transformations. We define the moduli space of Spin(7)-instantons as follows

$$\mathfrak{M} = \{A \in \mathcal{A} \mid \pi_7(F_A) = 0\} / \mathcal{G}.$$

As we are working with Sobolev norms to have more control of the topologies involved, we introduce the spaces:

$$\mathfrak{M}_s = \{A \in \mathcal{A}_s \mid \pi_7(F_A) = 0\} / \mathcal{G}_{s+1} = \{A \in \mathcal{A}_s \mid \pi_7(F_A) = 0\} / \bar{\mathcal{G}}_{s+1},$$

as well as the subspace of irreducible Spin(7)-instantons

$$\mathfrak{M}_s^* = \mathfrak{M}_s \cap \mathcal{B}_s^* = \{A \in \mathcal{A}_s^* \mid \pi_7(F_A) = 0\} / \mathcal{G}_{s+1} = \mathfrak{M}_s \cap \bar{\mathcal{B}}_s^* = \{A \in \mathcal{A}_s^* \mid \pi_7(F_A) = 0\} / \bar{\mathcal{G}}_{s+1}$$

Remark 5.4. We equip \mathfrak{M}_s and \mathfrak{M}_s^* with the subspace topologies, inherited from \mathcal{B}_s .

Under suitable conditions, we will obtain that $\bar{\mathfrak{M}}_s^*$ is a smooth finite-dimensional manifold.

Definition 5.5. We define the following linear operator of infinite-dimensional Hilbert spaces

$$(19) \quad \begin{aligned} L_A : \Omega_s^1(\mathfrak{g}_E) &\rightarrow \Omega_{s-1}^0(\mathfrak{g}_E) \oplus \Omega_{7,s-1}^2(\mathfrak{g}_E), \\ \tau &\mapsto d_A^* \tau \oplus \pi_7(d_A \tau). \end{aligned}$$

Lemma 5.6. *The linear operator (19) is elliptic.*

Proof. By proposition 3.7 the symbol of L_A is equal to the symbol of the Dirac operator associated to the Ivanov connection of the underlying Spin(7)-structure coupled to the connection induced by A on the endomorphism bundle. Hence the result follows. \square

Proposition 5.7. *Let $A \in \mathcal{A}_s$ be a Spin(7)-instanton. Then A is gauge equivalent to a smooth connection. Furthermore, for any $s > \frac{1}{2} \dim(M)$ we have the identification $\mathfrak{M} \cong \mathfrak{M}_s$ and \mathfrak{M} naturally becomes a second-countable, Hausdorff, metrizable topological space.*

Proof. First we prove that if $A_I \in \mathcal{A}_s$ is a Spin(7)-instanton, then there exists a gauge transformation $u \in \mathcal{G}_{s+1}$ such that $u \cdot A \in \mathcal{A}$, that is, $u \cdot A$ is smooth. Let then A_I be a Spin(7)-instanton. Since smooth connections are dense in \mathcal{A}_s when equipped with the H_s -topology, for every $\delta > 0$ there exists a smooth connection A_S such that $\|A_I - A_S\|_s < \delta$. We apply now theorem 4.20 to A_S , obtaining the existence of a $\epsilon_{A_S} > 0$ such that for any connection $A \in \mathcal{A}_s$ satisfying $\|A - A_S\|_s < \epsilon_{A_S}$ there exists a unique gauge transformation $u \in \mathcal{G}_{s+1}$ such that

$$u \cdot A = A_S + \tau, \quad d_A^* \tau = 0, \quad \|\tau\|_s < \epsilon_{A_S}.$$

We will take ϵ_{A_S} to be the supremum positive real number for which theorem 4.20 applies. We claim now that for every $A \in \mathcal{A}_s$, and in particular for A_I , there exists a smooth connection A_S such that $\|A - A_S\|_s < \epsilon_{A_S}$. To prove this we use the fact that \mathcal{A}_s is a metric space and in particular Fréchet-Urysohn. Therefore, there exists a sequence $\{A_S^n\}$ of smooth connections converging to A in the H_s -topology, i.e.

$$\lim_{n \rightarrow \infty} \|A_S^n - A\|_s = 0.$$

In addition, the sequence $\{A_S^n\}$ implies the existence of a sequence of positive numbers $\{\epsilon_{A_S^n}\} \subset \mathbb{R}^+$. We must have now $\epsilon_{A_S^n} > \|A_S^n - A\|_s$ for at least one $n = n_0 \in \mathbb{N}$. Otherwise $\epsilon_{A_S^n} \leq \|A_S^n - A\|_s$ for all n and hence $\lim_{n \rightarrow \infty} \epsilon_{A_S^n} = 0$, which implies, since $\{A_S^n\}$ converges to A , that there is no $\epsilon > 0$ satisfying theorem 4.20 when applied to A , and hence there is no slice $T_{A,\epsilon}$ around A , whence running into a contradiction with theorem 4.20 and hence proving the initial claim, since $\epsilon_{A_S^{n_0}} > \|A_S^{n_0} - A\|_s$ for at least one $n_0 \in \mathbb{N}$.

Let then A_S be a smooth connection such that $\|A_I - A_S\| < \epsilon_{A_S}$. By theorem 4.20, there exists a unique gauge transformation $u \in \mathcal{G}_{s+1}$ such that

$$u \cdot A_I = A_S + \tau, \quad d_{A_S}^* \tau = 0, \quad \|\tau\|_s < \epsilon_{A_S}.$$

Since A_I is a $\text{Spin}(7)$ -instanton and $F_{A_I} = F_{A_S} + d_{A_S} \tau + \frac{1}{2}[\tau, \tau]$, it follows that τ satisfies the following equation

$$\pi_7(F_{A_S}) + \pi_7\left(d_{A_S} \tau + \frac{1}{2}[\tau, \tau]\right) = 0.$$

We can rewrite the previous equation together with gauge-fixing condition as follows

$$L_{A_S}(\tau) + \left(0, \frac{1}{2}[\tau, \tau]\right) = (0, -\pi_7(F_{A_S})),$$

where $L_{A_S}: \Omega_s^1(\mathfrak{g}_E) \rightarrow \Omega_{s-1}^0(\mathfrak{g}_E) \oplus \Omega_{7,s-1}^2(\mathfrak{g}_E)$ is the linear operator of infinite-dimensional Hilbert spaces defined in equation (19), which by lemma 5.6 is elliptic. Clearly F_{A_S} is smooth and by the Sobolev multiplication theorem the term $\frac{1}{2}[\tau, \tau] \in \Omega_s^2(\mathfrak{g}_E)$ is in H_s . Therefore, applying the regularity theorem for elliptic operators on Sobolev spaces to the equation above we conclude that $\tau \in \Omega_{s+1}^1(\mathfrak{g}_E)$. Repeating the argument, $\tau \in \Omega_{s+k}^1(\mathfrak{g}_E)$, for all $k > 0$, and hence τ is smooth. This implies that $u \cdot A_I$ is a smooth connection.

Let us now consider the map

$$\mathbf{i}: \mathfrak{M} \rightarrow \mathfrak{M}_s,$$

where the topology of \mathfrak{M} is induced by the Fréchet topolgy of \mathcal{A} given by the C^∞ -convergence. In particular, \mathbf{i} is continuous. The previous argument shows that the map is surjective. We see that it is injective as follows: let A_1, A_2 be two smooth $\text{Spin}(7)$ -instantons, and suppose that $\mathbf{i}([A_1]) = \mathbf{i}([A_2])$ in \mathcal{B}_s . Then there exists $u \in \mathcal{G}_{s+1}$ such that $A_2 = u \cdot A_1$. By Lemma 4.13, we have that $u \in \mathcal{G}_{s+k}$ for all $k > 0$. Hence u is smooth and $[A_1] = [A_2]$ in \mathfrak{M} . Finally, let us see that \mathbf{i} is a closed map. Suppose that there is a sequence $[A_n]$ in \mathfrak{M} and $[A] \in \mathfrak{M}$ such that $\mathbf{i}([A_n]) \rightarrow \mathbf{i}([A])$, we have to see that $[A_n] \rightarrow [A]$ in \mathfrak{M} . Consider the slice $T_{A,\epsilon}$, then there exist some $u_n \in \mathcal{G}_{s+1}$ such that $u_n \cdot A_n \rightarrow A$ in H_s , $u_n \cdot A_n = A + \tau_n$, $d_A^* \tau_n = 0$, and $\|\tau_n\|_s \rightarrow 0$. Therefore, as before

$$L_A(\tau_n) + \left(0, \frac{1}{2}[\tau_n, \tau_n]\right) = (0, 0),$$

since $\pi_7(F_A) = 0$. Then inductively, $\tau_n \in \Omega_{s+k}^1(\mathfrak{g}_E)$ and so τ_n is C^∞ . Now decompose $\tau_n = \tau_n^0 + \tau_n^\perp$ according to the decomposition $\ker d_A^* = \mathbb{H}_A^1 \times \ker(\pi_7 \circ d_A)$. Let $c > 0$ be the first non-zero eigenvalue of L_A . Then

$$\|\tau_n^\perp\|_{s+k+1} \leq c^{-1} \|L_A(\tau_n^\perp)\|_{s+k} = c^{-1} \|L_A(\tau_n)\|_{s+k} \leq C \|\tau_n\|_{s+k}^2,$$

where $C > 0$ is a constant independent of $s+k$. Moreover all norms $\|\cdot\|_s$ on \mathbb{H}_A^1 are equivalent since this is a finite-dimensional vector space. So if $\|\tau_n\|_s \rightarrow 0$ then $\|\tau_n^0\|_{s+k} \rightarrow 0$. Also by induction on $k > 0$ and the above inequality, $\|\tau_n^\perp\|_{s+k} \rightarrow 0$. Then $\|\tau_n\|_{s+k} \rightarrow 0$, for all $k > 0$. Thus $\tau_n \rightarrow 0$ in the C^∞ -topology and $[A_n] \rightarrow [A]$ in \mathfrak{M} . \square

We define now the π_7 -projection of the curvature operator

$$\mathcal{F}_{7,s} := \pi_7 \circ \mathcal{F}_s: \mathcal{A}_s \rightarrow \Omega_{7,s-1}^2(\mathfrak{g}_E),$$

which is a smooth, bounded, \mathcal{G}_{s+1} -equivariant map of infinite-dimensional Hilbert-manifolds. The preimage of zero under $\mathcal{F}_{7,s}$ is the space of $\text{Spin}(7)$ -instantons on E . We clearly have

$$\mathfrak{M}_s = (\mathcal{F}_{7,s})^{-1}(0)/\bar{\mathcal{G}}_{s+1}.$$

For simplicity we define, for every $\tau \in \Omega^1(\mathfrak{g}_E)$, $\Psi_A(\tau) := \mathcal{F}_{7,s}(A + \tau)$. Hence

$$\begin{aligned} \Psi_A: \Omega_s^1(\mathfrak{g}_E) &\rightarrow \Omega_{7,s-1}^2(\mathfrak{g}_E), \\ \tau &\mapsto \pi_7 \left(d_A \tau + \frac{1}{2} [\tau, \tau] \right). \end{aligned}$$

We are interested in characterizing the local geometry of \mathfrak{M}_s . Consider $T_{A,\epsilon}$ as given in (12). We define the following restriction of Ψ_A

$$\Psi_{A,\epsilon} := \Psi_A|_{T_{A,\epsilon}}: T_{A,\epsilon} \rightarrow \Omega_{7,s-1}^2(\mathfrak{g}_E).$$

In addition we define

$$Z(\Psi_{A,\epsilon}) := \Psi_{A,\epsilon}^{-1}(0) \subset T_{A,\epsilon}.$$

Remark 5.8. If $A \in \mathcal{A}^*$, then there exists $\epsilon > 0$ such that $T_{A,\epsilon} \subset \mathcal{A}^*$. In that case $Z(\Psi_{A,\epsilon}) \subset \mathfrak{M}_s^*$.

Remark 5.9. The space $Z(\Psi_{A,\epsilon})$ can be explicitly defined as

$$Z(\Psi_{A,\epsilon}) := \{A + \tau, \tau \in \Omega_s^1(\mathfrak{g}_E) \mid d_A^* \tau = 0, \Psi_A(\tau) = 0, \|\tau\|_s < \epsilon\}.$$

Since Ψ_A is not a linear map, $Z(\Psi_{A,\epsilon})$ is a closed subset of $T_{A,\epsilon}$ which is not a linear subspace.

The following proposition follows from theorem 4.20 by restricting the homeomorphism h using the instanton condition.

Proposition 5.10. *For sufficiently small $\epsilon > 0$, the homeomorphism $h: T_{A,\epsilon}/\Gamma_A \rightarrow U([A]) \subset \mathcal{B}_s$ introduced in theorem 4.20 induces a homeomorphism*

$$h: Z(\Psi_{A,\epsilon})/\Gamma_A \rightarrow U([A]) \cap \mathfrak{M}_s \subset \mathfrak{M}_s.$$

In particular, \mathfrak{M}^* is locally homeomorphic to $Z(\Psi_{A,\epsilon})$ for every $A \in \mathcal{A}_s^*$.

In order to proceed further we need to examine in more detail the zero set $Z(\Psi_{A,\epsilon})$ and the local map $\Psi_{A,\epsilon}$. In particular, we will show that $\Psi_{A,\epsilon}$ is Fredholm on $T_{A,\epsilon}$, a fact which will give us a local model for the moduli space of Spin(7)-instantons in terms of the appropriate cohomology groups.

Let us recall that by definition $\Psi_{A,\epsilon}$ is Fredholm on $T_{A,\epsilon}$ if and only if, at every point in $\tau_0 \in T_{A,\epsilon}$, the derivative $D_{\tau_0} \Psi_{A,\epsilon}$ is a Fredholm linear operator of Hilbert spaces. In our case, this derivative is independent of τ_0 so we will drop the subscript. It is given by

$$\begin{aligned} D(\Psi_{A,\epsilon}): \ker d_A^* &\rightarrow \Omega_{7,s-1}^2(\mathfrak{g}_E), \\ \tau &\mapsto \pi_7(d_A \tau). \end{aligned} \tag{20}$$

Lemma 5.11. *Let $A \in \mathcal{A}$ be a Spin(7)-instanton. Then $D(\Psi_{A,\epsilon}): \ker d_A^* \rightarrow \Omega_{7,s-1}^2(\mathfrak{g}_E)$ is a Fredholm linear operator.*

Proof. Let us consider the sequence

$$0 \rightarrow \Omega_{s+1}^0(\mathfrak{g}_E) \xrightarrow{d_A} \Omega_s^1(\mathfrak{g}_E) \xrightarrow{\pi_7 \circ d_A} \Omega_{7,s-1}^2(\mathfrak{g}_E) \rightarrow 0, \tag{21}$$

which in addition is a complex since A is an Spin(7)-instanton. The associated symbol complex is a complex of vector bundles and linear maps given by

$$0 \rightarrow p^*(\Lambda^0(M) \otimes \mathfrak{g}_E) \xrightarrow{\delta_1} p^*(\Lambda^1(M) \otimes \mathfrak{g}_E) \xrightarrow{\delta_2} p^*(\Lambda^2(M) \otimes \mathfrak{g}_E) \rightarrow 0, \tag{22}$$

where we denote by $p: T^*M \rightarrow M$ the corresponding projection. The linear maps δ_1 and δ_2 evaluated at the cotangent vector $x \in T^*M$ are given by

$$\delta_1|_x = (\wedge x) \otimes \text{Id}, \quad \delta_2|_x = \pi_7 \circ ((\wedge x) \otimes \text{Id}).$$

Using the previous expressions for δ_1 and δ_2 , exactness of (22) follows by direct computation. Therefore the images of the complex (21) are closed subspaces of the corresponding Hilbert spaces and the associated cohomology groups are finite-dimensional. In addition we obtain

$$\Omega_s^1(\mathfrak{g}_E) = d_A(\Omega_{s+1}^0(\mathfrak{g}_E)) \oplus \ker d_A^* \tag{23}$$

which implies that the kernel and cokernel of (20) respectively correspond to the first and second cohomology groups of the complex (21). Since (21) is elliptic they are finite-dimensional. \square

Remark 5.12. Lemma 5.11 also follows from ellipticity of the linear operator

$$\begin{aligned} L_A: \Omega_s^1(\mathfrak{g}_E) &\rightarrow \Omega_{s-1}^0(\mathfrak{g}_E) \oplus \Omega_{7,s-1}^2(\mathfrak{g}_E), \\ \tau &\mapsto d_A^* \tau \oplus \pi_\tau(d_A \tau), \end{aligned}$$

defined in (19), see lemma 5.6.

Remark 5.13. Let us rewrite the deformation of the Spin(7)-instanton equation in terms of spinors given in (18). Suppose that $A \in \mathcal{A}$ satisfies $F_A \cdot \eta = 0$, and let $A' = A + \tau$ with $\tau \in \Omega^1(\mathfrak{g}_E)$. Then $F_{A'} \cdot \eta = 0$ is equivalent to $(d_A \tau + \frac{1}{2}[\tau, \tau]) \cdot \eta = 0$. The linearization of this equation is given by

$$d_A \tau \cdot \eta = 0,$$

which has to be supplemented with the gauge-fixing condition $d_A^* \tau = 0$.

Let ∇^T be the spin Ivanov connection associated to the Spin(7)-structure, which is the unique connection with fully antisymmetric torsion T such that $\nabla^T \eta = 0$. We follow now the idea proposed in [5] applied to our particular case. Associated to the Ivanov connection ∇^T on the spinor bundle and to the connection induced by A on \mathfrak{g}_E we consider the corresponding Dirac operator $D_T^-: \Omega^0(S^- \otimes \mathfrak{g}_E) \rightarrow \Omega^0(S^+ \otimes \mathfrak{g}_E)$. Then for $\tau \in \Omega^1(\mathfrak{g}_E)$ we have that

$$(24) \quad d_A \tau \cdot \eta = 0, d_A^* \tau = 0 \quad \text{if and only if} \quad D_T^-(\tau \cdot \eta) = 0,$$

as it happens in [5]. We define the following map

$$\begin{aligned} Q_A: \Omega^1(\mathfrak{g}_E) &\rightarrow \Omega^0(\mathfrak{g}_E) \oplus \Omega^0(H \otimes \mathfrak{g}_E), \\ \tau &\mapsto d_A^* \tau \oplus d_A \tau \cdot \eta, \end{aligned}$$

where $S^+ = \langle \eta \rangle \oplus H$ orthogonally. Thus $Q_A(\tau) = 0$ if and only if equations (24) hold. Using the isomorphism (2), we have $\Omega^0(H \otimes \mathfrak{g}_E) \cong \Omega_7^2(\mathfrak{g}_E)$, and we can write Q_A as

$$\begin{aligned} Q_A: \Omega^1(\mathfrak{g}_E) &\rightarrow \Omega^0(\mathfrak{g}_E) \oplus \Omega_7^2(\mathfrak{g}_E), \\ \tau &\mapsto d_A^* \tau \oplus \pi_\tau(d_A \tau), \end{aligned}$$

which coincides with L_A defined in equation (19). So, as expected, the infinitesimal deformations of the spinorial Spin(7)-instanton condition (18) are equivalent to the infinitesimal deformations of the Spin(7)-instanton equation.

The cohomology groups of the complex (21) are given by

$$\begin{aligned} \mathbb{H}_A^0 &= \ker d_A = \text{coker } L_A \cap \Omega_{s+1}^0(\mathfrak{g}_E), \\ \mathbb{H}_A^1 &= \frac{\ker(\pi_7 \circ d_A)}{\text{im}(d_A)} = \ker L_A \cap \Omega_s^1(\mathfrak{g}_E), \\ \mathbb{H}_A^2 &= \frac{\Omega_{7,s-1}^2(\mathfrak{g}_E)}{\text{im}(\pi_7 \circ d_A)} = \text{coker } L_A \cap \Omega_{s-1}^2(\mathfrak{g}_E). \end{aligned}$$

Here \mathbb{H}_A^0 is the space of infinitesimal automorphisms of A , \mathbb{H}_A^1 is the space of infinitesimal deformations of the Spin(7)-instanton A and \mathbb{H}_A^2 is the space of infinitesimal obstructions. If a connection A is irreducible then $\mathbb{H}_A^0 = 0$. We say that A is *regular* if $\mathbb{H}_A^2 = 0$.

Proposition 5.14. *The index of L_A is given by*

$$\text{index}(L_A) = -(\dim \mathfrak{g}) \hat{A}_2(M) + \frac{1}{24} \langle p_1(M), p_1(\mathfrak{g}_E) \rangle - \frac{1}{12} (p_1(\mathfrak{g}_E)^2 - 2p_2(\mathfrak{g}_E)).$$

where $\hat{A}_2(M) = \frac{7p_1(M)^2 - 4p_2(M)}{5670}$ is the second term of the \hat{A} -genus of M .

Proof. Proposition 3.7 shows that

$$\text{index}(L_A) = \text{index}(D_T^-) = \text{index}(D^-),$$

where $D^- : \Omega^0(S^- \otimes \mathfrak{g}_E) \rightarrow \Omega^0(S^+ \otimes \mathfrak{g}_E)$ is the Dirac operator associated to the Levi-Cevita spin connection and the connection induced by A in \mathfrak{g}_E . In eight dimensions the chiral complex spin bundles $S_{\mathbb{C}}^{\pm}$ are the complexifications of the real chiral spin bundles S^{\pm} . Let $D_{\mathbb{C}}^- : \Omega^0(S_{\mathbb{C}}^- \otimes (\mathfrak{g}_E \otimes \mathbb{C})) \rightarrow \Omega^0(S_{\mathbb{C}}^+ \otimes (\mathfrak{g}_E \otimes \mathbb{C}))$ denote the \mathbb{C} -linear extension of D^- . We have

$$\text{index}(D^-) = \text{index}_{\mathbb{C}}(D_{\mathbb{C}}^-) = - \int_M \hat{A}(M) \text{ch}(\mathfrak{g}_E \otimes \mathbb{C}),$$

where we have used the index theorem of the chiral complex spin bundle of a $8k$ -dimensional manifold coupled to the complexification $(\mathfrak{g}_E \otimes \mathbb{C})$ of \mathfrak{g}_E . Expanding now $\hat{A}(M)$ and $\text{ch}(\mathfrak{g}_E \otimes \mathbb{C})$ in terms of the Pontrjagin numbers of M and \mathfrak{g}_E we obtain

$$\text{index } L_A = -(\dim \mathfrak{g}) \hat{A}_2(M) + \frac{1}{24} \langle p_1(M), p_1(\mathfrak{g}_E) \rangle - \frac{1}{12} (p_1(\mathfrak{g}_E)^2 - 2p_2(\mathfrak{g}_E)).$$

□

For a non-integrable Spin(7)-manifold (M, Ω) , we define

$$(25) \quad b_7^2 = \frac{7p_1(M)^2 - 4p_2(M)}{5670} - 1 + b^1.$$

so $\hat{A}_2(M) = 1 - b_1 + b_7^2$ and hence the index of L_A is formally equal to the index in the Spin(7)-holonomy case, although only in the latter case b^1 and b_7^2 have the interpretation of the Betti numbers of M .

Theorem 5.15. *Let $A \in \mathfrak{M}_s$. Then there exist a neighborhood $U([A])$ of $[A]$ and a Γ_A -equivariant map*

$$f : \mathbb{H}_A^1 \rightarrow \mathbb{H}_A^2,$$

such that $U([A])$ is homeomorphic to $f^{-1}(0)/\Gamma_A$.

Proof. We have defined the map

$$\Psi_{A,\epsilon} : T_{A,\epsilon} \rightarrow \Omega_{7,s-1}^2(\mathfrak{g}_E),$$

whose zero level set $Z(\Psi_A)$ modulo Γ_A gives a local model of \mathfrak{M}_{s+1} around $[A] \in \mathfrak{M}_{s+1}$. Its differential $D\Psi_{A,\epsilon} : \ker d_A^* \rightarrow \Omega_{7,s-1}^2(\mathfrak{g}_E)$ at every point in $T_{A,\epsilon}$, in particular at $0 \in T_{A,\epsilon}$, is Fredholm and has closed range. Therefore, there are decompositions

$$\ker d_A^* = \ker(\pi_7 \circ d_A) \times \mathbb{H}_A^1, \quad \Omega_{7,s-1}^2(\mathfrak{g}_E) = \ker(\pi_7 \circ d_A)^* \times \mathbb{H}_A^2,$$

in terms of the hypercohomology groups defined in (25). By the Open Mapping theorem $\Psi_{A,\epsilon}|_{V_0} : V_0 = \ker(\pi_7 \circ d_A) \rightarrow W_0 = \ker(\pi_7 \circ d_A)^*$ is a Hilbert-space isomorphism. Using now Fredholm theory and the fact that $\Psi_{A,\epsilon}(0) = 0$ we conclude that there exist neighborhoods $U(0) \subset T_{A,\epsilon}$ and $V(0) \subset \Omega_{7,s-1}^2(\mathfrak{g}_E)$ of zero such that

- $U(0) = U_1 \times U_2$ and $V(0) = V_1 \times V_2$ with $U_2 \subset \mathbb{H}_A^1$ and $V_2 \subset \mathbb{H}_A^2$ are (necessarily finite-dimensional) neighborhoods of zero.
- For every $u := (u_1, u_2) \in U_1 \times U_2$ we have

$$(26) \quad \Psi_{A,\epsilon}(u_1, u_2) = (F_1(u_1), F_2(u_1, u_2)),$$

where $F_1 : U_1 \rightarrow V_1$ is a diffeomorphism of Hilbert manifolds and $F_2 : U_1 \times U_2 \rightarrow V_2$ is a differentiable map of Hilbert manifolds.

Using equation (26) we can characterize $\Psi_{A,\epsilon}^{-1}(0)$ as the pre-image of zero by the differentiable map

$$\begin{aligned} f : U_2 \subset \mathbb{H}_A^1 &\rightarrow V_2 \subset \mathbb{H}_A^2, \\ u_2 &\mapsto F_2(0, u_2), \end{aligned}$$

and hence we conclude. □

Remark 5.16. For an irreducible connection $[A] \in \mathfrak{M}_s^*$ we conclude that there exists a neighborhood $U([A])$ of $[A]$ homeomorphic to $f^{-1}(0)$. If in addition $\mathbb{H}_A^2 = 0$ we deduce that \mathfrak{M}_s^* is locally modelled on the vector \mathbb{H}_A^1 and hence we conclude that \mathfrak{M}_s^* is a smooth manifold. In section 6 we are going to study how generic is the situation for which $\mathbb{H}_A^2 = 0$ in terms of a generic choice of $\text{Spin}(7)$ -structure.

The case $\mathfrak{g}_E = \mathfrak{u}_E$ with fixed determinant. In this section we consider the particular case $\mathfrak{g}_E = \mathfrak{u}_E$, fixing in addition a connection on the determinant line bundle. Let $E \rightarrow M$ be a hermitian complex vector bundle of rank r , over an 8-manifold M endowed with a $\text{Spin}(7)$ -structure given by a four-form Ω . The structure group of the vector bundle is $G = U(r)$. As explained in section 2, we have a decomposition $\Lambda^2 = \Lambda_7^2 \oplus \Lambda_{21}^2$ with projections $\pi_7 : \Lambda^2 \rightarrow \Lambda_7^2$ and $\pi_{21} : \Lambda^2 \rightarrow \Lambda_{21}^2$. Let $L = \det E$ be the determinant line bundle, where we fix a connection Λ . Note that there is a decomposition $\mathfrak{u}_E = \mathbb{R} \oplus \mathfrak{su}_E$, where the \mathbb{R} -summand correspond to the trace of the connection. The space of connections with fixed determinant is

$$\mathcal{A}^\Lambda = \{A \in \mathcal{A} \mid \text{tr } A = \Lambda\}.$$

Fixing $A_0 \in \mathcal{A}^\Lambda$, any other connection $A = A_0 + \tau$ has $\tau \in \Omega^1(\mathfrak{su}_E)$. Therefore $\mathcal{A}^\Lambda = A_0 + \Omega^1(\mathfrak{su}_E)$. Let $\alpha \in \Omega^1$ be the curvature of Λ . Any connection $A \in \mathcal{A}^\Lambda$ has curvature splitting as $F_A = (\text{tr } F_A)\text{Id} + F_A^0$, where F_A^0 is the trace-free part of the curvature. We have that $\text{tr } F_A = \alpha$. So the curvature is defined by the variable part $F_A^0 \in \Omega^2(\mathfrak{su}_E)$. We give the following definition

Definition 5.17. [43] A connection $A \in \mathcal{A}^\Lambda$ is a $\text{Spin}(7)$ -instanton if $\pi_7(F_A^0) = 0$.

Finally, we consider the gauge group

$$\mathcal{G}^\Lambda = \{g \in \mathcal{G} \mid \det g = \text{Id}\},$$

which acts on \mathcal{A}^Λ , since the action on the determinant line bundle $L = \det E$ is trivial, and hence it does not move the connection. There is a space of connections modulo gauge

$$\mathcal{B}^\Lambda = \mathcal{A}^\Lambda / \mathcal{G}^\Lambda,$$

and a moduli space of $\text{Spin}(7)$ -instantons $\mathfrak{M}^\Lambda = \{[A] \mid \pi_7(F_A^0) = 0\}$.

As in the general case we Sobolev-complete \mathcal{A}^Λ as to obtain \mathcal{A}_s^Λ , as well as the other spaces. Given $A \in \mathcal{A}_s^\Lambda$ a slice around A for the action of $\mathcal{G}_{s+1}^\Lambda$ is given by

$$T_{A,\epsilon} := \{A + \tau \mid \tau \in \Omega_s^1(\mathfrak{su}_E), d_A^* \tau = 0, \|\tau\|_s < \epsilon\} \subset \mathcal{A}_s^\Lambda.$$

Let $A \in \mathcal{A}_s^\Lambda$ be a $\text{Spin}(7)$ -instanton and let $A' = A + \tau \in \mathcal{A}_s^\Lambda$ be another connection, $\tau \in \Omega_s^1(\mathfrak{su}_E)$. Then

$$\pi_7(F_{A'}^0) = \pi_7\left(d_A \tau + \frac{1}{2}[\tau, \tau]\right).$$

Therefore $A + \tau$ satisfies the $\text{Spin}(7)$ -instanton equation if and only if $\pi_7(d_A \tau + \frac{1}{2}[\tau, \tau]) = 0$. So the theory proceeds now exactly as in the general case, with \mathfrak{su}_E playing the role of \mathfrak{g}_E . There is a deformation complex

$$(27) \quad 0 \rightarrow \Omega_{s+1}^0(\mathfrak{su}_E) \xrightarrow{d_A} \Omega_s^1(\mathfrak{su}_E) \xrightarrow{\pi_7 \circ d_A} \Omega_{7,s-1}^2(\mathfrak{su}_E) \rightarrow 0,$$

with the operator

$$\begin{aligned} L_A : \Omega_s^1(\mathfrak{su}_E) &\rightarrow \Omega_{s-1}^0(\mathfrak{su}_E) \oplus \Omega_{7,s-1}^2(\mathfrak{su}_E), \\ \tau &\mapsto d_A^* \tau \oplus \pi_7(d_A \tau). \end{aligned}$$

Proposition 5.14 about the index of L_A can be refined for the special case considered in this section.

Proposition 5.18. *The index of L_A is given by*

$$\begin{aligned} \text{index } L_A = & -(r^2 - 1)(1 - b_1 + b_2^2) + \frac{1}{24} \langle p_1(M), -2r c_2(E) + (r - 1)c_1(E)^2 \rangle \\ & - \left(\frac{r+7}{12} c_1(E)^4 - \frac{r+6}{3} c_1(E)^2 c_2(E) + \frac{r+3}{3} c_1(E) c_3(E) + \frac{r+6}{6} c_2(E)^2 - \frac{r}{3} c_4(E) \right). \end{aligned}$$

Proof. In the particular case of $\mathfrak{g} = \mathfrak{su}(r)$, a Chern class calculation gives:

$$\begin{aligned} p_1(\mathfrak{g}_E) &= -2r c_2(E) + (r-1)c_1(E)^2, \\ \frac{1}{12}(p_1(\mathfrak{g}_E)^2 - 2p_2(\mathfrak{g}_E)) &= \text{ch}_4(\mathfrak{g}_E) = \text{ch}_4(\text{End } E) = \\ &= \frac{r+7}{12}c_1(E)^4 - \frac{r+6}{3}c_1(E)^2c_2(E) + \frac{r+3}{3}c_1(E)c_3(E) + \frac{r+6}{6}c_2(E)^2 - \frac{r}{3}c_4(E). \end{aligned}$$

We substitute in the formula in proposition 5.14. \square

Remark 5.19. The formula in proposition 5.18 specializes to that of reference [44] when $c_1 = 0$. Note that there is a mistake in a sign in (2.25) in [44].

Remark 5.20. Proposition 5.18 specializes for the case of rank $r = 2$ bundles to

$$-3(1 - b_1 + b_2^7) + \frac{1}{24}\langle p_1(M), -4c_2(E) + c_1(E)^2 \rangle - \left(\frac{3}{4}c_1(E)^4 - \frac{8}{3}c_1(E)^2c_2(E) + \frac{4}{3}c_2(E)^2 \right).$$

6. TRANSVERSALITY OF THE MODULI SPACE OF Spin(7)-INSTANTONS

Let (M, Ω) be a Spin(7)-manifold, and let E be an rank r complex vector bundle endowed with an hermitian metric. Fix a connection Λ on the determinant line bundle $L = \det E$ of E , and let $A \in \mathcal{A}^\Lambda$ be a Spin(7)-instanton. Associated to it, we have a deformation complex (27). We denote by $\mathbb{H}_A^0, \mathbb{H}_A^1, \mathbb{H}_A^2$ its hypercohomology groups. Recall that if A is irreducible then $\mathbb{H}_A^0 = 0$. We say that A is regular if $\mathbb{H}_A^2 = 0$. By theorem 5.15, if A is irreducible and regular, then \mathfrak{M}_s^* is a smooth manifold around A , of finite dimension. In general, regularity does not hold. In this section, we shall study in detail how to perturb the equations to get regularity.

Let $\mathcal{S}(M) := \Omega^0(\Sigma(M))$ be the space of Spin(7)-structures on M , namely the space of smooth sections of $\Sigma(M) \subset \Lambda^4(M)$. We shall consider tensors of type C^k , for some large k , and give $\mathcal{S}(M)$ the C^k -topology, so that it becomes a Banach manifold. Associated to each $\Omega \in \mathcal{S}(M)$ there is a projector P_Ω :

$$\begin{aligned} P_\Omega : \Omega_{s-1}^2(\mathfrak{su}_E) &\rightarrow \Omega_{P_\Omega, s-1}^2(\mathfrak{su}_E) \\ \beta &\mapsto \frac{1}{2}(\beta + *_\Omega(\Omega \wedge \beta)), \end{aligned}$$

where $\Omega_{P_\Omega, s-1}^2(\mathfrak{su}_E)$ denotes the space of forms taking values in \mathfrak{su}_E of type $\Omega_{P_\Omega, s-1}^2(\mathfrak{su}_E)$ with respect to Ω . Let $\Omega_0 \in \mathcal{S}(M)$ be a fixed Spin(7)-structure, and $P_0 = P_{\Omega_0}$ the associated projector. For Ω near Ω_0 , the projection

$$P_0 : \Omega_{P_\Omega, s-1}^2(\mathfrak{su}_E) \rightarrow \Omega_{P_0, s-1}^2(\mathfrak{su}_E)$$

is an isomorphism. Therefore, the equations

$$P_\Omega(F_A^0) = 0 \iff P_0(P_\Omega(F_A^0)) = 0$$

are equivalent, but the second one has a fixed target space. We now consider the map

$$\begin{aligned} \mathcal{L} : \mathcal{A}_s \times \mathcal{S}(M) &\longrightarrow \Omega_{P_0, s-1}^2(\mathfrak{su}_E) \\ (A, \Omega) &\mapsto P_0(P_\Omega(F_A^0)) \end{aligned}$$

which corresponds to a parametric version of the Spin(7)-instanton equation.

To prove that the moduli space for some $\Omega \in \mathcal{S}(M)$ is regular at any connection A with $\mathcal{L}(A, \Omega) = 0$, we need to prove that $D_1 \mathcal{L}$ is surjective, where D_1 denotes the differential with respect to the first variable. The general set up is as follows: suppose $\mathcal{U}, \mathcal{S}, \mathcal{W}$ are Banach manifolds and $F : \mathcal{U} \times \mathcal{S} \rightarrow \mathcal{W}$ is a smooth map such that $F_s : \mathcal{U} \rightarrow \mathcal{W}$, $F_s(x) = F(x, s)$ is Fredholm for any $s \in \mathcal{S}$. Suppose that F is a submersion over a point $0 \in \mathcal{W}$, that is $D_{(x,s)}F : T_x \mathcal{U} \times T_s \mathcal{S} \rightarrow T_0 \mathcal{W}$ is onto for $F(x, s) = 0$.

Then $\mathcal{M} = F^{-1}(0) \subset \mathcal{U} \times \mathcal{S}$ is a smooth Banach manifold. The projection $\Pi : \mathcal{U} \times \mathcal{S} \rightarrow \mathcal{S}$ restricts to a smooth map $\Pi|_{\mathcal{M}} : \mathcal{M} \rightarrow \mathcal{S}$, and

$$\begin{aligned} \ker(D_{(x,s)}\Pi|_{\mathcal{M}}) &\cong \ker D_x F_s, \\ \text{coker}(D_{(x,s)}\Pi|_{\mathcal{M}}) &\cong \text{coker } D_x F_s. \end{aligned}$$

So $\Pi|_{\mathcal{M}}$ is a smooth Fredholm map between Banach manifolds. Recall now the Sard-Smale theorem.

Theorem 6.1 ([39]). *Let $\phi : \mathcal{M} \rightarrow \mathcal{S}$ be a C^q -Fredholm map between separable Banach manifolds, with $q > 0, q > \text{index } \phi$. Then the set of regular values of ϕ is residual in \mathcal{S} . In particular, it is dense.*

Now consider a regular value $s_0 \in \mathcal{S}$ for $\Pi|_{\mathcal{M}}$. Then $\text{coker } D_x F_{s_0} = \text{coker}(D_{(x,s_0)}\Pi|_{\mathcal{M}}) = 0$, for any (x, s_0) such that $F_{s_0}(x) = F(x, s_0) = 0$. This means that $F_{s_0} : \mathcal{U} \rightarrow \mathcal{W}$ is regular over the point $0 \in \mathcal{W}$, and $F_{s_0}^{-1}(0) \subset \mathcal{U}$ is a smooth manifold.

Thus, going back to our original problem, if we prove that $D_1 \mathcal{L}$ is surjective whenever $\mathcal{L}(A, \Omega) = 0$, then we have the required transversality for a dense set of $\Omega' \in \mathcal{S}(M)$.

Proposition 6.2. *Let $\Omega_0 \in \mathcal{S}(M)$, and let $A \in \mathcal{A}_s$ be a $\text{Spin}(7)$ -instanton with respect to Ω_0 . Let $\psi \in \Omega_{7,s-1}^2(\mathfrak{su}_E)$ be L^2 -orthogonal to the image of $D_{(A,\Omega_0)}\mathcal{L}$. Then $d_A^* \psi = 0$ and $\text{tr}(F_A^0 \wedge \psi) = 0$.*

Proof. Let $\mathcal{P} : \mathcal{S}(M) \rightarrow \text{Hom}(\Omega_{s-1}^2(M), \Omega_{s-1}^2(M))$ be the map $\mathcal{P}(\Omega) = P_\Omega$. Then the map $\mathcal{L}(\Omega, A) = P_0(P(F_A^0))$ has linearization

$$(28) \quad D_{(A,\Omega_0)}\mathcal{L}(a, \omega) = P_0(d_A a) + P_0(D_{\Omega_0}\mathcal{P}(\omega)(F_A^0)).$$

Take $\psi \in \Omega_{P_0, s-1}^2(\mathfrak{su}_E)$ orthogonal to $D_{(A,\Omega_0)}\mathcal{L}(a, \omega)$ as given in equation (28). Then

$$\begin{aligned} \langle P_0(d_A a), \psi \rangle &= 0, \quad \text{for all } a \in \Omega_s^1(\mathfrak{su}_E), \\ \langle P_0(D_{\Omega_0}\mathcal{P}(\omega)(F_A^0)), \psi \rangle &= 0, \quad \text{for all } \omega \in T_{\Omega_0}\mathcal{S}(M). \end{aligned}$$

The first equation is rewritten $\langle d_A a, \psi \rangle = 0$, since ψ is already in $\Omega_{P_0, s-1}^2(\mathfrak{su}_E)$. Equivalently $\langle a, d_A^* \psi \rangle = 0$, for all $a \in \Omega_s^1(\mathfrak{su}_E)$. This means that

$$d_A^* \psi = 0.$$

The second equation is rewritten as $\langle D_{\Omega_0}\mathcal{P}(\omega)(F_A^0), \psi \rangle = 0$. To simplify it, consider the formula

$$\int_M g_\Omega(*_\Omega(\Omega \wedge F_A^0), \psi) = \int_M \Omega \wedge \text{tr}(F_A^0 \wedge \psi),$$

which holds for all Ω , where g_Ω is the scalar product induced by Ω . We take its derivative at Ω_0 in the direction of ω , and recall that $P_\Omega(\beta) = \frac{1}{2}(\beta + *_\Omega(\Omega \wedge \beta))$, so

$$\langle D_{\Omega_0}\mathcal{P}(\omega)(F_A^0), \psi \rangle + \int_M D_{\Omega_0}\mathcal{G}(\omega)(P_0(F_A^0)), \psi = \int_M \omega \wedge \text{tr}(F_A^0 \wedge \psi),$$

where $\mathcal{G} : \mathcal{S}(M) \rightarrow \text{Met}(M)$ is the map $\mathcal{G}(\Omega) = g_\Omega$. As A is a $\text{Spin}(7)$ -instanton, we have that $P_0(F_A^0) = 0$. Thus the second equation is rewritten as $\int_M \omega \wedge \text{tr}(F_A^0 \wedge \psi) = 0$, for all $\omega \in T_{\Omega_0}\mathcal{S}(M)$. By proposition 2.3, $T_{\Omega}\mathcal{S}(M) = \Omega_1^4(M) \oplus \Omega_7^4(M) \oplus \Omega_{35}^4(M)$. Therefore $\text{tr}(F_A \wedge \psi) \in \Lambda_{27}^4(M)$. On the other hand, $F_A \in \Lambda_{21}^2(\mathfrak{su}_E)$ and $\psi \in \Lambda_7^2(\mathfrak{su}_E)$. By proposition 2.1, $\text{tr}(F_A \wedge \psi) \in (\Lambda_7^4(M) \oplus \Lambda_{35}^4(M))$. This means that $\text{tr}(F_A \wedge \psi) = 0$. \square

Now we shall take more general type of perturbations. Fix a background $\text{Spin}(7)$ -structure Ω_0 , and therefore also a corresponding metric. Let $\mathcal{P}(M)$ be the set of all orthogonal projectors $P : \Lambda^2(M) \rightarrow \Lambda^2(M)$ of rank-seven. As before, we consider tensors of class C^k , for suitable large k , so that $\mathcal{P}(M)$ becomes a Banach manifold. We consider the projector $P_0 \in \mathcal{P}(M)$ associated to Ω_0 , and the perturbed equation

$$P(F_A^0) = 0,$$

for $A \in \mathcal{A}_s^\Lambda$. Let

$$\mathfrak{M}_s^P = \{A \in \mathcal{A}_s^\Lambda \mid P(F_A^0) = 0\} / \mathcal{G}_{s+1}$$

be the perturbed moduli space. We shall consider \mathfrak{M}_s^P only for P near P_0 . Note that in this case

$$P(F_A^0) = 0 \iff P_0(P(F_A^0)) = 0,$$

and the second equation takes values in a fixed space $\Omega_{P_0, s-1}^2(\mathfrak{su}_E)$. Consider the functional

$$\begin{aligned} \mathcal{F}_P : \mathcal{A}_s^\Lambda &\rightarrow \Omega_{P, s-1}^2(\mathfrak{su}_E), \\ A &\mapsto P(F_A^0) \end{aligned}$$

and the functional $\mathcal{F}_P^0(A) = P_0(P(F_A^0)) : \mathcal{A}_s^\Lambda \rightarrow \Omega_{P_0, s-1}^2(\mathfrak{su}_E)$. We want to prove that the moduli space \mathfrak{M}_s^P is regular at an irreducible connection A . This means the surjectivity of $D_A \mathcal{F}_P$ at any A with $\mathcal{F}_P(A) = 0$, which in turn is equivalent to the surjectivity of $D_A \mathcal{F}_P^0$.

As before, we consider the parametric version of the perturbed equation, given by the map

$$\begin{aligned} \mathcal{L} : \mathcal{A}_s \times \mathcal{P}(M) &\longrightarrow \Omega_{P_0, s-1}^2(\mathfrak{su}_E) \\ (A, P) &\mapsto P_0(P(F_A^0)). \end{aligned}$$

To prove that the moduli space \mathfrak{M}_s^P , for some $P \in \mathcal{P}(M)$, is regular at any irreducible connection A with $\mathcal{L}(A, P) = 0$, we need to prove that $D_1 \mathcal{L}$ is surjective, where D_1 denotes the differential with respect to the first variable. If that case holds then, by our previous argument, for a dense second category subset of P near P_0 (in the topology of $\mathcal{P}(M)$), we will have that $D_A \mathcal{F}_P^0$ is surjective, for generic P , since $\mathcal{L}(A, P) = \mathcal{F}_P^0(A)$. This will complete the required transversality.

Proposition 6.3. *Let $\Omega_0 \in \mathcal{S}(M)$, $P_0 = P_{\Omega_0}$, and let $A \in \mathcal{A}_s^\Lambda$ be a Spin(7)-instanton with respect to Ω_0 . Let $\psi \in \Omega_{7, s-1}^2(\mathfrak{su}_E)$ be L^2 -orthogonal to the image of $D_{(A, P_0)} \mathcal{L}$. Then $d_A^* \psi = 0$ and $\text{tr}(F_A^0 \otimes \psi) = 0$, as a section of $\Lambda_{21}^2(M) \otimes \Lambda_7^2(M)$.*

Proof. We fix one projector P_0 and a decomposition $\Lambda^2 = \Lambda_7^2 \oplus \Lambda_{21}^2$. Other decompositions correspond to the graph of a map

$$\mu : \Lambda_7^2 \longrightarrow \Lambda_{21}^2.$$

Here the projector is

$$P = \begin{pmatrix} p & p\mu^* \\ \mu p & \mu p \mu^* \end{pmatrix},$$

where $p = (1 + \mu\mu^*)^{-1}$. We consider small perturbations around $\mu = 0$, given by some $\dot{\mu}$. We have

$$\dot{P} = \begin{pmatrix} 0 & \nu^* \\ \nu & \nu + \nu^* \end{pmatrix}.$$

The derivative of the map $\mathcal{L}(A, P) = P_0(P(F_A^0))$ is given by

$$D_{(A, P_0)} \mathcal{L}(a, \nu) = P_0(d_A a) + P_0(\dot{P}(F_A^0)) = P_0(d_A a) + \nu^*(F_A^0),$$

where $F_A^0 \in \Omega_{21, s-1}^2(\mathfrak{su}_E)$. Let now $\psi \in \Omega_{7, s-1}^2(\mathfrak{su}_E)$ be an element orthogonal to the image of $D_{(A, P_0)} \mathcal{L}$. This implies that

$$\begin{aligned} \langle P_0(d_A a), \psi \rangle &= 0, \quad \forall a \in \Omega^1(\mathfrak{su}_E), \\ \langle \nu^*(F_A^0), \psi \rangle &= 0, \quad \forall \nu \in \Omega^0(\text{Hom}(\Lambda_7^2, \Lambda_{21}^2)). \end{aligned}$$

The first equation yields that $d_A^* \psi = 0$. The second equation is equivalent to

$$\int_M \text{tr}(\nu^*(F_A^0), \psi) = \langle \nu, \text{tr}(F_A^0 \otimes \psi) \rangle = 0,$$

for all $\dot{\mu}$, considered as a section of $\Lambda_{21}^2 \otimes \Lambda_7^2$. Therefore

$$\text{tr}(F_A \otimes \psi) = 0.$$

□

The perturbations $P \in \mathcal{P}(M)$ allow to obtain transversality for the moduli spaces \mathfrak{M}_s^P in the specific case that $E \rightarrow M$ is a rank 2 vector bundle.

Theorem 6.4. *Suppose that E is a rank 2 vector bundle. Let $A \in \mathcal{A}_s^\Lambda$ be a $\text{Spin}(7)$ -instanton. Let $\psi \in \Omega_7^2(\mathfrak{su}_E)$ such that $d_A^* \psi = 0$ and $\text{tr}(F_A^0 \otimes \psi) = 0$. Then A is reducible or $\psi = 0$.*

Proof. The equation $\text{tr}(F_A^0 \otimes \psi) = 0$, where $F_A^0 \in \Omega_{21}^2(\mathfrak{su}_E)$ and $\psi \in \Omega_7^2(\mathfrak{su}_E)$, means that the (bundle) maps

$$F_A^0 : \Lambda_{21}^2 \longrightarrow \mathfrak{su}_E, \quad \psi : \Lambda_7^2 \longrightarrow \mathfrak{su}_E$$

have images which are point-wise orthogonal. As $\mathfrak{su}(2)$ has dimension 3, this implies that either F_A^0 is a map of rank ≤ 1 or ψ is a map of rank ≤ 1 , at any point of M .

Suppose first that $F_A^0 = 0$ in a ball. Recall that $d_A F_A^0 = 0$, by the Bianchi identity. As $*F_A^0 = -\Omega \wedge F_A^0$, we have that

$$d_A^* F_A^0 = - * (W \wedge F_A^0),$$

where $W = d\Omega$. Then F_A^0 satisfies an elliptic equation, and hence $F_A^0 = 0$ everywhere. This means that A is projectively flat, in particular it is not irreducible according to our definition.

Now suppose that F_A^0 has rank 1 in a ball. We trivialize the bundle over the ball, and let $e_1 \in \mathfrak{su}(2)$ be a unit vector spanning the image of F_A^0 . We complete to an orthonormal basis $\{e_1, e_2, e_3\}$, and write $F_A^0 = \omega \otimes e_1$, where $\omega \in \Omega_{21}^2$. Write $d_A e_1 = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$, where $\lambda_j \in \Omega^1$. From $d_A F_A^0 = 0$ we get

$$0 = d_A F_A^0 = (d\omega + \omega \wedge \lambda_1) \otimes e_1 + \omega \wedge \lambda_2 \otimes e_2 + \omega \wedge \lambda_3 \otimes e_3.$$

Hence $\omega \wedge \lambda_2 = 0$ and $\omega \wedge \lambda_3 = 0$. If ω is a 2-form and λ is a non-zero 1-form with $\omega \wedge \lambda = 0$, then $\omega = \lambda \wedge \Theta$ for a 1-form Θ . But then it cannot be that $\omega \in \Lambda_{21}^2$. Therefore $\lambda_2 = \lambda_3 = 0$. This implies that $d_A e_1 = \lambda_1 e_1$. As e_1 is unitary, we have that $d_A e_1 = 0$. So A is locally reducible, because we can split $\mathfrak{su}_E = \langle e_1 \rangle \oplus \langle e_2, e_3 \rangle$, and A respect both summands.

Next suppose that ψ has rank 1 in a ball. Write $\psi = \omega \otimes e_1$, where $\omega \in \Lambda_7^2$, and $\{e_1, e_2, e_3\}$ is a local orthonormal basis of \mathfrak{su}_E . Write $d_A e_1 = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$. We use the equation $d_A^* \psi = 0$ and the equality $*\psi = 3\Omega \wedge \psi$ to get $d_A(\Omega \wedge \psi) = 0$. So

$$0 = d_A(\Omega \wedge \omega \otimes e_1) = (d(\Omega \wedge \omega) + \Omega \wedge \omega \wedge \lambda_1) \otimes e_1 + \Omega \wedge \omega \wedge \lambda_2 \otimes e_2 + \Omega \wedge \omega \wedge \lambda_3 \otimes e_3.$$

In particular, $\Omega \wedge \omega \wedge \lambda_2 = 0$ and $\Omega \wedge \omega \wedge \lambda_3 = 0$. The map $\Omega : \Lambda^3 \rightarrow \Lambda^7$ has kernel Λ_{38}^3 , so we have $(\omega \wedge \lambda_2)_8 = 0$, where this is the component in Λ_8^3 . But for any element $\omega \in \Lambda_7^2$, the map

$$\omega : \Lambda_8^1 \rightarrow \Lambda_8^3$$

is an isomorphism; being a map of $\text{Spin}(7)$ -representations, it is equivalent to Clifford multiplication $V \otimes H \rightarrow S^-$, by our discussion of section 2. Therefore $\lambda_2 = \lambda_3 = 0$. As argued before, we conclude that A is locally reducible.

Finally if $\psi = 0$ on a ball, then using that $d_A^* \psi = 0$ and $*\psi = 3\Omega \wedge \psi$, we get that

$$d_A \psi = 3 * (W \wedge \psi).$$

So ψ satisfies an elliptic equation and $\psi = 0$ everywhere, which is one of the possibilities in the statement of the theorem.

Assume that $\psi \neq 0$. Let us see that the set U of points $x \in M$ such that A is reducible on a ball B around x is dense. Let $x_0 \in M$. If either $\text{rk } \psi(x_0) = 1$ or $\text{rk } F_A^0(x_0) = 1$, then A is reducible on a ball around x_0 as argued above. As either $\text{rk } \psi(x_0) \leq 1$ or $\text{rk } F_A^0(x_0) \leq 1$, then we may assume that one of them vanishes on x_0 , say $\psi(x_0) = 0$. It cannot be $\psi = 0$ on a ball B around x_0 . If there is a point $x \in B$ with $\text{rk } \psi(x) = 1$, we have $x \in U$, as required. If not, then the set of points with $\{x \in B \mid \text{rk } \psi(x) = 2\}$ is open and dense in a suitable small ball around x . At those points $\text{rk } F_A^0(x) \leq 1$, but it cannot be that $F_A^0(x) = 0$ on all of them, because it is an open set. So there must be some point with $\text{rk } F_A^0(x) = 1$, and this proves that $x \in U$. Once we have that U is dense, the result follows from the proposition 6.5 below. \square

Proposition 6.5. *Let A be a $\text{Spin}(7)$ -instanton which is locally reducible. Then A is reducible.*

Proof. Let A be a connection and suppose that it is reducible on a ball. Then the connection is of the form

$$A = \begin{pmatrix} ai & 0 \\ 0 & -ai \end{pmatrix}, \quad a \in \Omega^1.$$

The connection on \mathfrak{su}_E is of the form

$$(29) \quad A = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2a \\ 0 & -2a & 0 \end{pmatrix}.$$

If $a = 0$ in an open subset, then $F_A^0 = 0$ everywhere, as argued in the proof of theorem 6.4. This would conclude that A is reducible. Otherwise $a \neq 0$ on an dense subset of the ball, and then e_1 is uniquely determined over the ball (up to sign). We assume this henceforth.

At every point x where A is locally reducible, there is a unique u such that $\nabla_A u = 0$. Let $R > 0$ be the injectivity radius of M , and take a ball B of radius R around x . Using geodesic coordinates and parallel transport along radial geodesics, we trivialize the bundle E and the connection A . We take a basis of \mathfrak{su}_E at x so that $u = e_1$. The connection is given by a 1-form on B of the form

$$A = \begin{pmatrix} 0 & -\beta & -\gamma \\ \beta & 0 & -\alpha \\ \gamma & \alpha & 0 \end{pmatrix}.$$

The local reducibility gives that $A \wedge A = 0$ on a dense subset and hence everywhere. On a neighbourhood $B' \subset B$ of x we have that $\nabla_A e_1 = 0$. This is equivalent to $\beta = \gamma = 0$, or $A = \alpha \otimes e_1$, that is, equal to (29). Also note that $F_A = d\alpha \otimes e_1$. If we prove that $\beta = \gamma = 0$ on the whole of B , then A is reducible over the larger ball B . This happens at every point x , and by density, we will have that there is some section $u \in \Gamma(\mathfrak{su}_E)$ with $\nabla_A u = 0$, proving reducibility on the whole manifold M .

Suppose that there are points $y \in B$ such that $A = \delta \otimes f$, for some 1-form δ and unitary f with $\nabla_A f = 0$. Note that $\nabla_A f = df$, so f must be constant on our trivialization, that is $f \in \mathfrak{su}(2)$. The connection A is C^∞ meaning that α, δ are C^∞ , when extending them by zero outside the locus where A is in the direction of e_1, f respectively. If we consider the closure of the set where $F_A = d\alpha \otimes e_1$ and the closure of the set where $F_A = d\delta \otimes f$, then in the intersection we have $F_A = 0$ and $A = 0$.

So let V be the set where A is locally reducible. Suppose that it has different connected components, and take the connected component W that contains B' . Over W , there is a parallel section u , and A can be written as $A = \alpha \otimes u$, at least locally. The curvature has a global expression $F_A = d\alpha \otimes u$ over there. That is, $F_A = \omega \otimes u$, where ω is a closed 2-form. This form can be extended by zero to the complement of W , and it is C^∞ . Then

$$\begin{aligned} d\omega &= 0, \\ d^*\omega &= \omega \wedge W, \end{aligned}$$

where the first equality follows from the Bianchi identity, and the second one since $*\omega = -(\omega \wedge \Omega)$, because A is a Spin(7)-instanton.

Therefore ω satisfies an elliptic equation. If there are other components appart from W , then ω vanishes in some open set in the complement of W . So by elliptic regularity it should be $\omega = 0$. This is a contradiction and completes the proof of the proposition. \square

Theorem 6.4 implies that for generic P near P_0 , the irreducible locus $\mathfrak{M}^{P,*}$ is a smooth manifold of finite dimension given by the index in remark 5.20. To argue this, first note that we may take P a C^∞ -projector, since these are dense in the given topology. Consider the equation

$$(30) \quad \begin{aligned} \mathcal{F}_P : \mathcal{A}_s^\Lambda &\rightarrow \Omega_{P,s-1}^2(\mathfrak{su}_E), \\ A &\mapsto P(F_A^0). \end{aligned}$$

For $A \in \mathcal{A}_s$ satisfying $\mathcal{F}_P(A) = 0$, we take a slice $T_{A,\epsilon}$ given by the gauge fixing condition $d_A^* \tau = 0$, $\|\tau\|_s < \epsilon$, where $A' = A + \tau$. This gives a functional

$$\begin{aligned} L_A^P : \Omega_s^1(\mathfrak{su}_E) &\rightarrow \Omega_{s-1}^0(\mathfrak{su}_E) \times \Omega_{P,s-1}^2(\mathfrak{su}_E), \\ \tau &\mapsto (d_A^* \tau, d_A^P \tau), \end{aligned}$$

where $d_A^P = P \circ d_A$. This map L_A^P is Fredholm for P near P_0 , since the Fredholm condition is open. The index of L_A^P is the same as that of L_A (given in proposition 5.18) by the same reason. Our previous arguments work verbatim for the equation (30). Therefore, if L_A^P is surjective then \mathfrak{M}_s^P is a smooth manifold of finite dimension around $[A]$.

Finally, proposition 5.7 can also be carried out for the case of (30), giving that the moduli space

$$\mathfrak{M}^P = \{A \in \mathcal{A}^\Lambda \mid \mathcal{F}_P(A) = 0\} / \mathcal{G}^\Lambda$$

is homeomorphic to $\mathfrak{M}^P \cong \mathfrak{M}_s^P$. Therefore \mathfrak{M}^P is a second-countable, Hausdorff, metrizable topological space which has the structure of a smooth manifold on the irreducible locus.

Theorem 6.6. *For a dense family of projector perturbations P the moduli spaces \mathfrak{M}^P are smooth at irreducible connections, of dimension given by remark 5.20. They are second-countable, Hausdorff and metrizable.*

Holonomy perturbations. Now we want to give a different type of perturbation that allows to deal with higher rank bundles. These are called *holonomy perturbations* are well-known in the context of instantons on 4-dimensional manifolds [29].

Let (M, Ω) be an 8-dimensional manifold endowed with a $\text{Spin}(7)$ -structure (not necessarily integrable). Let G be a semi-simple compact Lie group, with Lie algebra \mathfrak{g} , and let $P \rightarrow M$ be a principal G -bundle. As before, consider a faithful representation of G and the associated complex vector bundle $E \rightarrow M$. Let \mathcal{A} be the space of G -connections on E . We perturb the $\text{Spin}(7)$ -instanton equation

$$\mathcal{F} : \mathcal{A} \longrightarrow \Omega_7^2(\mathfrak{g}_E), \quad \mathcal{F}(A) = F_A$$

as follows. Consider tuples $(x, \gamma, B, h, \omega)$ where $x \in M$, γ is a loop based at x , B is a small ball around x , $h : B \times [0, 1] \rightarrow M$ is a smooth map with $h|_{B \times \{t\}}$ an embedding of B to a ball centered at $\gamma(t)$, and $h|_{B \times \{0\}} = h|_{B \times \{1\}} = \text{Id}$, and ω is a 2-form on B , with compact support and lying in $\Omega_7^2(\mathfrak{g}_E)$. For each tuple as above, we define a map

$$V_{h,\omega} : \mathcal{A} \longrightarrow \Omega_7^2(\mathfrak{g}_E).$$

as follows. For $A \in \mathcal{A}$, and for each $y \in B$, consider the holonomy around $\gamma_y(t) = h(y, t)$, $h_A(y) = \text{hol}_{\gamma_y}(A) \in \text{Ad } P_y \cong G \subset \text{End } E_y$. This defines a section h_A of $\text{Ad}(P)$ over B . Fix an embedding $\text{Ad}(P) \cong G \subset M_{n \times n}(\mathbb{C})$, and then project orthogonally to $\mathfrak{g} \subset M_{n \times n}(\mathbb{C})$, obtaining a section of \mathfrak{g}_E over B . Multiplying by ω , we have an element

$$V_{h,\omega}(A) = \omega \cdot h_A \in \Omega_7^2(\mathfrak{g}_E).$$

Take a collection of points (x_n) dense in M . For each x_n , consider a collection of loops (γ_m) dense in the space of loops in M based at x_n , in the C^1 -topology. Using a diagonal procedure, we obtain a collection (x_n, γ_n) of such elements. For each n and s , we consider some ω_n as before with C^s -norm bounded by some $c_{n,s} > 0$, and we require

$$\|c\|_s := \sum_{n=1}^{\infty} c_{n,s} < \infty.$$

The perturbation parameter is $\Theta = \{(x_n, \gamma_n, B_n, h_n, \omega_n) \mid n \in \mathbb{N}\}$, and we denote the space of such perturbation parameters as \mathcal{W} . Note that this is a Fréchet space. Define

$$V_\Theta(A) = \sum_{n=1}^{\infty} V_{h_n, \omega_n}(A).$$

This gives a well-defined map

$$V_\Theta : \mathcal{A} \longrightarrow \Omega_7^2(\mathfrak{g}_E).$$

Now we perturb the Spin(7)-instanton equations, considering

$$P_\Theta : \mathcal{A} \longrightarrow \Omega_7^2(\mathfrak{g}_E), \quad P_\Theta(A) = F_A^7 + V_\Theta(A),$$

We define the moduli space of *perturbed Spin(7)-instantons* as

$$\mathfrak{M}^\Theta = \{A \in \mathcal{A} \mid P_\Theta(A) = 0\} / \mathcal{G}.$$

The study of the topological properties of \mathfrak{M}^Θ is similar to that of \mathfrak{M} . To prove smoothness of the irreducible locus of \mathfrak{M}^Θ , we need to study the deformation complex

$$\Omega^0(\mathfrak{g}_E) \xrightarrow{d_A} \Omega^1(\mathfrak{g}_E) \xrightarrow{d_A^\Theta} \Omega_7^2(\mathfrak{g}_E),$$

where $d_A^\Theta(a) = d_A^7 a + D_A V_\Theta(a)$ is the linearization of P_Θ . To study it, we consider a Sobolev norm H_s and the corresponding gauge group \mathcal{G}_{s+1} , and space of connections \mathcal{A}_s . For a tuple $(x, \gamma, B, h, \omega)$, the map $V_{h,\omega}$ extends to a smooth map of Banach spaces [29, Prop. 3.1],

$$V_{h,\omega} : \mathcal{A}_s \longrightarrow L_s^2(\Lambda_7^2 \otimes \mathfrak{g}_E),$$

which satisfies that

$$\|V_{h,\omega}(A)\| \leq \|\omega\|_{C^0}.$$

and

$$\|D_A V_{h,\omega}(a)\|_s \leq K \|\omega\|_{C^s} \|a\|_s,$$

for some $K > 0$. Now we consider the space \mathcal{W}_s of perturbation parameters with $\|c\|_s < \infty$, for the given value of s . This is now a Banach space. The map V_Θ extends to

$$V_\Theta : \mathcal{A}_s \longrightarrow \Omega_{7,s}^2(\mathfrak{g}_E).$$

This produces a moduli space of H_s -*perturbed Spin(7)-instantons*

$$\mathfrak{M}_s^\Theta = \{A \in \mathcal{A}_s \mid P_\Theta(A) = 0\} / \mathcal{G}_{s+1},$$

with $P_\Theta(A) = F_A^7 + V_\Theta(A)$, as before. The deformation complex

$$\Omega_{s+1}^0(\mathfrak{g}_E) \xrightarrow{d_A} \Omega_s^1(\mathfrak{g}_E) \xrightarrow{d_A^\Theta} \Omega_{7,s-1}^2(\mathfrak{g}_E)$$

is elliptic, for a small perturbation parameter. The associated map

$$\begin{aligned} L_A^\Theta : \Omega_s^1(\mathfrak{g}_E) &\rightarrow \Omega_{s-1}^0(\mathfrak{g}_E) \oplus \Omega_{7,s-1}^2(\mathfrak{g}_E), \\ \tau &\mapsto (d_A^* \tau, d_A^7 \tau + D_A V_\Theta(\tau)), \end{aligned}$$

is Fredholm with index given by proposition 5.14.

We aim next to prove that for a dense set of parameters Θ , the map L_A^Θ is surjective. For this, we consider the parametric version

$$\mathcal{P} : \mathcal{W}_s \times \mathcal{A}_s \longrightarrow L_{s-1}^2(\Lambda_7^2 \otimes \mathfrak{g}_E).$$

We want to apply the Smale-Sard theorem 6.1. For this we need the differential at (A, Θ) ,

$$\begin{aligned} D_1 \mathcal{P}(a, \nu) &= d_A^7 a + D_A V_\Theta(a), \\ D_2 \mathcal{P}(a, \nu) &= V_\nu(A). \end{aligned}$$

As we have argued before, we only need to see that $D_2 \mathcal{P}$ is surjective at a point $(A, 0)$ with A an irreducible Spin(7)-instanton. Note that $D_1 \mathcal{P}$ is Fredholm, since it is the sum of an elliptic operator (which is Fredholm) and a compact operator (since $D_A V_\Theta$ is bounded from L_s^2 to L_s^2 , it is compact from L_s^2 to L_{s-1}^2). So the range of $D_1 \mathcal{P}$ is closed and of finite codimension.

Take $\psi \in \Omega_7^2(\mathfrak{g}_E)$ in the orthogonal complement of $D_1 \mathcal{P}$. Recall that we are assuming that A is irreducible. So for any x_n , the holonomies of γ_m based at x_n generate $\text{Ad } P_{x_n}$, since these loops are dense and the connection is irreducible. Therefore perturbing only one ν_n , we have that $\langle \nu \otimes h_A, \psi \rangle = 0$, for all ν . Hence $\psi(x_n) = 0$. By density of the x_n , we have that $\psi = 0$ everywhere.

This concludes that for a dense family of Θ , we have that L_A^Θ is surjective. Moreover, we may take $\Theta \in \mathcal{W}$, by density of \mathcal{W} in \mathcal{W}_s . In this case, we can prove, following the same argumental line as before, that $\mathfrak{M}^\Theta \cong \mathfrak{M}_s^\Theta$. Therefore we have proven that

Theorem 6.7. *For a dense family of holonomy perturbations Θ the moduli spaces \mathfrak{M}^Θ are smooth at irreducible connections, of dimension given by proposition 5.14. They are second-countable, Hausdorff and metrizable.*

7. Spin(7)-INSTANTONS FOR LINE BUNDLES

When requiring the fixed determinant condition for Spin(7)-instantons, the moduli space of Spin(7)-instantons on a line bundle is just a point. However, it is important to understand the space of solutions to the Spin(7)-instanton equation $P_\Omega(F_A) = 0$ on a line bundle without the fixed determinant condition, since they appear for reducible connections on higher rank bundles. For instance, if A is a reducible connection on a rank-two bundle E with fixed determinant $L = \det E$, then E splits as $E = L_1 \oplus (L \otimes L_1^*)$, and the connection A induces a Spin(7)-instanton on the line bundle L_1 .

Let M be a Spin(7)-manifold with a Spin(7)-structure given by a 4-form Ω . Let $L \rightarrow M$ be a U(1)-bundle, with Chern class $c_1(L) \in H^2(M, \mathbb{Z})$, whose image in real cohomology we denote by $c_1^\mathbb{R}(L) \in H^2(M, \mathbb{R})$. We want to find the moduli space of Spin(7)-instantons. Let A_0 be a connection on L , with curvature F_0 . Then $c_1^\mathbb{R}(L) = [\frac{F_0}{2\pi i}] \in H^2(M, \mathbb{R})$. Other connections $A = A_0 + ia$ are given by one-forms $a \in \Omega^1(M)$, so the space of connections is $\mathcal{A} = A_0 + i\Omega^1(M)$. The curvature is $F = F_0 + i da$. The Spin(7)-instanton equation is

$$P_\Omega(A) = \frac{1}{2}(F + *(F \wedge \Omega)) = 0.$$

The gauge group is simply given by $\mathcal{G} = \mathcal{C}^\infty(M, S^1)$. This has connected components parametrized by $[M, S^1] = H^1(M, \mathbb{Z})$. The connected component \mathcal{G}_0 of the identity is given by the maps $g = \exp(i\theta)$, $\theta \in \mathcal{C}^\infty(M)$. Hence $T_{\text{Id}}\mathcal{G}_0 = i\Omega^0(M)$. The action of \mathcal{G}_0 on \mathcal{A} is given by

$$g \cdot A = A + g^{-1}dg = A + i d\theta.$$

Therefore the orbit of the action of \mathcal{G}_0 on $A \in \mathcal{A}$ is given by $A + i \text{im } d$.

We complete all spaces of sections with a Sobolev norm H_s . As $\Omega_s^1(M) = \text{im } d \oplus \ker d^*$, we have that a *global* slice of the action is given by $\ker d^*$. Therefore the moduli space

$$\tilde{\mathfrak{M}}_L = \{A \in \mathcal{A} \mid P_\Omega(A) = 0\} / \mathcal{G}_0$$

is given by

$$\tilde{\mathfrak{M}}_L = \{A_0 + a, \ a \in \Omega^1(M) \mid L(a) = 0\},$$

where we have defined

$$(31) \quad \begin{aligned} L: \Omega_s^1(M) &\rightarrow \Omega_{s-1}^0(M) \oplus \Omega_{P_\Omega, s-1}^2(M), \\ a &\mapsto d^*a \oplus d^P a, \end{aligned}$$

and A_0 is a point in $\tilde{\mathfrak{M}}_L$. Note that the action of $\mathcal{G}/\mathcal{G}_0$ on any $A = A_0 + a$ is given by $a \mapsto a + \ell$, where $\ell \in \Omega^1(M)$ is an element of $\ker d^* \cap \ker d$, i.e., the harmonic representative of the class of $\mathcal{G}/\mathcal{G}_0 \cong [M, S^1] \cong H^1(M, \mathbb{Z})$. This implies that a b^1 -torus

$$A_0 + H^1(M, \mathbb{R})/H^1(M, \mathbb{Z}) \subset \mathfrak{M}_L = \{A \in \mathcal{A} \mid P(A) = 0\} / \mathcal{G}$$

sits in the moduli space of solutions, for any $A_0 \in \mathfrak{M}_L$. This follows since if a_0 is a solution of $L(a_0) = 0$, then $a_0 + u$ is also a solution for any u harmonic.

Integrable case. First let us suppose that the Spin(7)-holonomy is integrable. Then F is closed and $d^*F = *d*F = -*d(F \wedge \Omega) = 0$, since Ω is closed. So F is harmonic. This means that $F = F_0$, the harmonic representative of $c_1^{\mathbb{R}}(L)$. When M has holonomy Spin(7), we have a decomposition

$$H^2(M) = H_7^2(M) \oplus H_{21}^2(M),$$

where $H_7^2(M)$ is the space of harmonic forms β with $\beta = 3 * (\beta \wedge \Omega)$ and $H_{21}^2(M)$ is the space of harmonic forms β with $\beta = - * (\beta \wedge \Omega)$. Let b_7^2, b_{21}^2 be the dimensions of such spaces.

The deformation complex of \mathfrak{M}_L is

$$0 \rightarrow \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d^P} \Omega_{P\Omega}^2(M) \rightarrow 0,$$

with $d^P = P \circ d$. It has $\mathbb{H}^0 = \mathbb{R}$, $\mathbb{H}^1 = H^1(M)$, $\mathbb{H}^2 = H_7^2(M)$. The first statement is clear. The second follows from the fact that $\ker d^P = \ker d$. If $\beta \in \Omega^1(M)$ satisfies that $d^P \beta = 0$. Then $d\beta = - * (d\beta \wedge \Omega)$. This implies that $d\beta$ is closed and co-closed, hence harmonic. Therefore $d\beta = 0$. The third statement is proved as follows: any harmonic form $\gamma \in H_7^2(M)$ gives an element in $\Omega_{P\Omega}^2(M)$ with $d^* \gamma = 0$. Hence $\langle \gamma, d\beta \rangle = \langle \gamma, P(d\beta) \rangle = \langle \gamma, d^P \beta \rangle = 0$, for all $\beta \in \Omega^1(M)$. Hence $H_7^2(M) \rightarrow \mathbb{H}^2$ is injective. Now take an element $\gamma \in \mathbb{H}^2$. This means that $\gamma \in \Omega_7^2(M)$. We project into the orthogonal space to $\text{im } d^P$, which gives an element representing the same class and $\langle \gamma, d^P \beta \rangle = \langle \gamma, d\beta \rangle = 0$, for all β . Thus $d^* \gamma = 0$. As $*\gamma = -\gamma \wedge \Omega$, we have that γ is closed, hence harmonic, so it lives in $H_7^2(M)$.

Therefore if $c_1^{\mathbb{R}}(L) \in H_{21}^2(M)$, then there is a connection with harmonic curvature in $\Omega_{21}^2(M)$. The space of solutions is given by connections with harmonic curvature, hence

$$\mathfrak{M}_L = H^1(M, \mathbb{R})/H^1(M, \mathbb{Z}).$$

If $c_1^{\mathbb{R}}(L) \notin H_{21}^2(M)$, then there is no harmonic element representing $c_1^{\mathbb{R}}(L)$ in $\Omega_{21}^2(M)$, hence $\mathfrak{M}_L = \emptyset$. Note that when $b_7^2 > 0$, we have $\mathbb{H}^2 \neq 0$, so the solutions to the Spin(7)-equations are not regular. This fact, together with the fact that $\dim \mathbb{H}^0 = 1$ is clearly reflected in the explicit value of the virtual dimension of \mathfrak{M}_L , given by minus the index of L ,

$$\text{index}(L) = 1 - b^1 + b_7^2.$$

The virtual dimension $-\text{index}(L)$ differs thus from the real dimension b^1 precisely by the dimension of the vector space $\mathbb{H}^0 \oplus \mathbb{H}^2$, as expected.

If $c_1^{\mathbb{R}}(L) = 0$, namely if L is torsion, then the moduli space of Spin(7)-instantons is always non-empty and we have

$$\mathfrak{M}_L = H^1(M, \mathbb{R})/H^1(M, \mathbb{Z}) = \text{Hom}(\pi_1(M), U(1)),$$

and hence every Spin(7)-instanton is in this case a flat connection.

Non-integrable case. Now suppose that Ω is a non-closed Spin(7)-structure, that is, $W = d\Omega$ is possibly non-zero. Let A_0 be a connection on L with curvature F_0 , which represents $c_1^{\mathbb{R}}(L)$. The moduli space \mathfrak{M}_L is given by

$$\tilde{\mathfrak{M}}_L = \{A_0 + a, a \in \Omega^1(M) \mid L(a) = 0\},$$

where we have defined

$$L_0: \Omega^1(M) \rightarrow \Omega^0(M) \oplus \Omega_{P\Omega}^2(M),$$

as follows

$$(32) \quad L_0(a) = (d^*a, P_\Omega(a)), \quad P_\Omega(a) = \frac{1}{2}(da + *(da \wedge \Omega)) + \beta_0,$$

with $\beta_0 = \frac{1}{2}(F_0 + *(F_0 \wedge \Omega))$. Consider the deformation complex

$$(33) \quad \Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d^P} \Omega_{P\Omega}^2(M),$$

with $d^P = P_\Omega \circ d$, and let $\mathbb{H}^0, \mathbb{H}^1, \mathbb{H}^2$ be its hypercohomology groups. In the non-integrable case, we define b_7^2 by the formula (25), so the index of the complex (33) equals $1 - b_1 + b_7^2$. In particular, as obviously $\dim \mathbb{H}^1 \geq b^1$ always, then $\dim \mathbb{H}^2 \geq b_7^2$.

Theorem 7.1. *Let Ω be a Spin(7)-form, and let $W = d\Omega$. Let $\lambda_1 > 0$ be the smallest non-zero eigenvalue of the Laplacian. If $\|W\| < \lambda_1$, then the space of solutions \mathfrak{M}_L , if non-empty, is $H^1(M, \mathbb{R})/H^1(M, \mathbb{Z})$.*

Proof. Suppose that A_0 is a Spin(7)-instanton with curvature F_0 . Then any the other solution $A \in \mathcal{A}$ to the Spin(7)-instanton equation can be written as $A = A_0 + a$, where $a \in \Omega^1(M)$ with $d^*a = 0$, and $da = - * (\Omega \wedge da)$. We have

$$\|da\|^2 = \int da \wedge *da = - \int da \wedge da \wedge \Omega = - \int a \wedge da \wedge W,$$

by integration by parts in the last equality. Suppose that a is orthogonal to the harmonic forms, and let us see that $da = 0$. We have that $a = G(\Delta a) = G(d^*da)$, where G is the Green's operator. Then

$$\|da\|^2 \leq \|a\| \|da\| \|W\|$$

and

$$\|a\| = \|G(d^*da)\| \leq \frac{1}{\lambda_1} \|d^*da\| \leq \frac{1}{\lambda_1} \|da\|.$$

Thus

$$\|da\|^2 \leq \frac{1}{\lambda_1} \|da\|^2 \|W\|.$$

So if $\|W\| < \lambda_1$, then $da = 0$ and hence every Spin(7)-instanton is of the form $A = A_0 + a$ with a harmonic. The result follows. \square

Theorem 7.2. *Under the conditions of theorem 7.1, we have, for generic Ω ,*

- If $b_7^2 > 0$, the moduli space is empty.
- If $b_7^2 = 0$ or $c_1^{\mathbb{R}}(L) = 0$, then the moduli space is $H^1(M, \mathbb{R})/H^1(M, \mathbb{Z})$.

Proof. Suppose first that $b_7^2 = 0$. Then the moduli space \mathfrak{M}_L is regular, and hence of dimension b^1 . Take F_0 a 2-form representing $c_1^{\mathbb{R}}(L)$. Then $\beta_0 = P_{\Omega}(F_0) \in \Omega_{P_{\Omega}}^2(M)$. By the surjectivity of d^P , there is some $a \in \Omega^1(M)$ with $P_{\Omega}(da) = \beta_0$. This gives a solution to (32), which moreover is unique up to an element of \mathbb{H}^1 . Thus $\mathfrak{M}_L \cong H^1(M, \mathbb{R})/H^1(M, \mathbb{Z})$.

Now suppose that $b_7^2 > 0$ and $c_1^{\mathbb{R}}(L) \neq 0$. First, let us see that we have regularity for a small perturbation of Ω . By proposition 6.2, that works exactly in the same way for the case of non-fixed determinant, we have that $\text{tr}(F_A \wedge \psi) = F_A \wedge \psi = 0$, where $F_A \in \Omega_{21}^2(M)$ and $\psi \in \Omega_7^2(M)$. Then using remark 2.2, we have that $\psi = 0$ since $F_A \neq 0$ (which follows from $c_1^{\mathbb{R}}(L) \neq 0$). This completes the claim. Now we have that for a generic nearby Ω , we have regularity of the moduli space \mathfrak{M}_L . But then the dimension should be $b^1 - b_7^2 < b^1$. Therefore theorem 7.1 implies that \mathfrak{M}_L is empty.

Finally, suppose that $c_1^{\mathbb{R}}(L) = 0$. Then we can choose $F_0 = 0$ and equation (32) has solutions. Theorem 7.1 gives the statement. \square

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